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New Pathways in Aeronautical Theory

Fifth Wright Brothers Lecture*

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Introduction

To be invited to deliver your Wright Brothers Lecture must at any time be reason for keen satisfaction, but in normal times would have set me the awkward question—am I competent to undertake the task? This year, when personal reasons tend to weight the scales, I confess that I have felt no hesitation—my desire to come was stronger than my doubts. You have done me great honor, and you have given me an opportunity, which every worker in aeronautics must covet, of paying my humble tribute to the authors of that great achievement now 38 years gone by. For both I am deeply grateful.

It has become customary, in recent years, to preface aeronautical lectures such as this with lachrymose ponderings, brief or extended, on the question whether it was a good gift or a bad that Wilbur and Orville Wright gave to the world in 1903. I, who could never see human flight as a thing unconditionally good, do not propose to follow that tradition. The airplane was not the first gift of science which mankind has been free to use or to misuse, and it will not be the last. All power is double-edged, and always (unless the inventive faculty atrophies) we shall have to face responsibility for the use we make of power. Let us ponder that responsibility, for it is both great and inescapable; but not in scientific lectures. There let us rather give thanks, as men of science, for the fascinating problems which aeronautics has brought us in its train.

A New Mathematics

I was permitted to choose my title, and indeed, only on those terms could I have undertaken to lecture; for a man must talk of that with which his mind is filled, and I confess that in the last six years my interests have been no longer general, my mind has been focused steadily and sharply on one single aim. That aim is the development of a new mathematics, a mathematics better suited than what is orthodox to the needs of engineering and aeronautical research. As a line of study it has proved most happily appropriate to wartime conditions, with their pressure of new and urgent problems; and it still keeps our small "team" fully and contentedly occupied. Not all of our results may be revealed, some are still covered by the veil of wartime secrecy, but what can be released may perhaps serve to attract your interest and even to persuade some research worker to try our methods for himself.

What then is my complaint against methods which are current? Chiefly this, that *they take no cognizance of the unavoidable uncertainty of physical data.* The physicists have their "uncertainty principle": we too, I submit, should have our own. A simple example will serve to illustrate my point. (See Fig. 1.)

Here is a "redundant framework," redundant even when we neglect altogether the stiffness of its joints as distinct from the stiffness of its members. Similar examples, but simpler, are expected by every candidate for examination in structural engineering, since the methods available for their solution are a commonplace of structural theory. What I want now to emphasize is a "bogus" feature of such problems as set in examinations, *a feature which orthodox methods of solution do nothing to discourage.* Only a very impractical examiner would give marks for purporting accuracy (of calculated stresses or deflexions) closer (say) than 1 per cent; the candidate, that is to say, may use a slide-rule. But the bogus feature is this, that the specified loading—5 tons here, 10 tons there—is specified as

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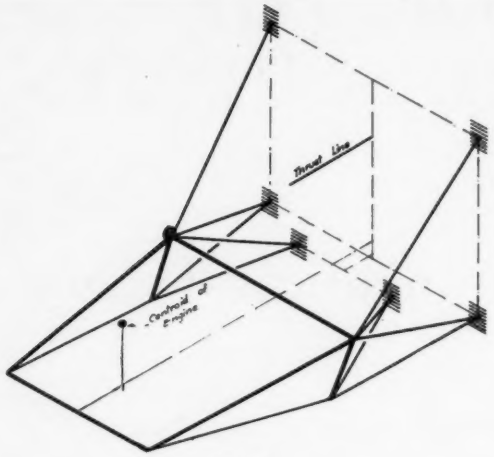


FIG. 1.

though the examiner knew it *exactly*—had it by supranatural revelation.

In fact, loadings are never known exactly. Usually aerodynamic or inertial, they are as uncertain as anything in the data of a problem, more uncertain (normally) than the scantlings of the members or the elastic properties of their materials. To them, if to anything in the data, "tolerances" should be attached: 5 ± 0.1 tons here, 10 ± 0.3 tons there. But this is rarely done, even in cases we confront as problems of research; *because the doing of it would have no consequences, in orthodox methods of attack.*

THE MARGIN OF UNCERTAINTY

Suppose that we do insert these tolerances—giving to every load the "margin of uncertainty" which it always has in practice: then, clearly, *any solution which accounts for the loads within their margins of uncertainty has the same claim as any other to be termed "correct."* "Exact" solution becomes a word of no meaning in a theoretical treatment, just as "exact" measurement has no meaning in experiment. Will not theory gain reality from this parallelism with the experimental outlook?

For some 6 years I have sought, with a zealous band of assistants, to explore the implications of this new outlook. Breaking away from orthodox procedure, *we have focused attention, always, not on the wanted quantities (stresses or displacements) but on the data of each problem, on the loads.* We assume these to have always some specified "margin of uncertainty," and we "whittle away" at them—accounting for more and more—until every "residual" of a datum quantity has been brought within that margin. At this point we stop, asserting that to "whittle" further would be not only waste of time, but *meaningless.* If you cannot tell me the loads exactly (and in practice you never can), then you have no reason to expect unique solutions. We need not apologize—we may even boast—that ours is "mathematics with a fringe."

Taking this standpoint—which I hold to be thus defensible philosophically—we have found our problems not made harder, but greatly simplified. This whittling process, if at times laborious, is easy: it could be taught to any intelligent bank-clerk. For it comes to this merely, that we give him a row of numbers (*e.g.*, loads) which have to be accounted for, and a table of "standard operations" (Table 1) whereby he can whittle them down to within a specified margin of uncertainty. Each operation, usually, is more powerful than any other to whittle some particular load: we indicate this in the "operations table," and when "whittling"—speaking generally—we go always for the biggest "residual" (the biggest load not yet accounted for). Errors need not be cumulative, and at the finish we have a complete picture of the wanted configuration—displacements, stresses, everything. Given time, we can work to any margin of uncertainty, however fine, so long as this is not zero. The wider the margin (*and this is left for the designer, not the computer, to assess*), the quicker our route to a solution. Table 1 shows a typical "Operations Table" and the "Relaxation Table" in which this is utilized.

Before the whittling process can begin, of course, we must construct the "operations table": that is our first job—and surprisingly easy. For consider what is involved (again in relation to the typical framework of Fig. 1). Various operations can be imagined, but the simplest—yet quite general—is the imposition of a chosen displacement on a particular joint, *all other joints being held fixed.* Here is a problem special to each particular structure and to each particular joint; but it too can be left to our bank-clerk, if we can give him a general formula for the effect of any displacement imposed at one end *of a single member* when the other end is held fixed (Fig. 2).

FIG. 2.

THE UNIT PROBLEM

So we come (for frameworks) to a "unit problem" soluble in general terms, and to formulas applicable without restriction. They are not greatly complicated when we take account of rigidity in the joints; and in their application redundancy—that bug-bear of conventional methods—no longer multiplies toil.

Pursuing these ideas in relation to continuous girders, we found our treatment to be one, in essence, with the Moment-Distribution Method of Prof. Hardy Cross.

TABLE 1

(Units: 1 ton weight; 1 foot.)

Operation no.	Nature of operation	(1) X_A	(2) X_B	(3) Y_A	(4) Y_B	(5) Y_C	X_C	X_D	Y_D
1 (*a*)	$u_A = 1$	*−3,980*	*0*	*312*	*0*	*768*	*1,280*	*2,700*	*−1,080*
1 (*b*)	= 0.2513	−1,000	0	78$_5$	0	193	322	678	−271$_5$
2 (*a*)	$u_B = 1$	*0*	*−4,100*	*0*	*720*	*0*	*3,380*	*720*	*−720*
2 (*b*)	= 0.2439	0	−1,000	0	175	0	825	175	−175
3 (*a*)	$v_A = 1$	*312*	*0*	*−5,393*	*4,500*	*461*	*768*	*−1,080*	*432*
3 (*b*)	= 0.1854	58	0	−1,000	834	85$_5$	142$_5$	−200$_5$	80$_5$
4 (*a*)	$v_B = 1$	*0*	*720*	*4,500*	*−5,220*	*0*	*0*	*−720*	*720*
4 (*b*)	= 0.1916	0	138	862	−1,000	0	0	−138	138
5 (*a*)	$v_C = 1$	*768*	*0*	*460.8*	*0*	*−2,480.8*	*−768*	*0*	*2,020*
5 (*b*)	= 0.4031	309$_5$	0	186	0	−1,000	−309$_5$	0	814
6 (*a*)	$v_A = v_B = 1$	*312*	*720*	*−893*	*−720*	*461*	*768*	*−1,800*	*1,152*
6 (*b*)	= 0.6200	193	446	−554	−446	286	476	−1,116	714
7 (*a*)	$v_A = -v_B = 1$	*312*	*−720*	*−9,893*	*9,720*	*461*	*768*	*−360*	*−288*
7 (*b*)	= 0.0510	16	−37	−504	496	23	39	−18	−15

Operation	Multiplier × 10^3	X_A	X_B	Y_A	Y_B	Y_C
	(Initial forces)→	15	7.07	..	7.07	..
1 (*b*)	15	−15	0	1.18	0	2.89
		0	7.07	1.18	7.07	2.89
2 (*b*)	7	0	−7.00	0	−1.22	0
		0	0.07	1.18	8.29	2.89
6 (*b*)	10	1.93	4.46	−5.54	−4.46	2.86
		1.93	4.53	−4.36	3.83	5.75
7 (*b*)	−8	−0.13	0.30	4.03	−3.97	−0.18
		1.80	4.83	−0.33	−0.14	5.57
5 (*b*)	7	2.18	0	1.30	0	−7.00
		3.98	4.83	0.97	−0.14	−1.43
2 (*b*)	6	0	−6.00	0	1.05	0
		3.98	−1.17	0.97	+0.91	−1.43
1 (*b*)	5	−5.00	0	0.39	0	+0.97
		−1.02	−1.17	1.36	+0.91	−0.46
6 (*b*)	2.5	0.48	+1.12	−1.39	−1.11	0.72
		−0.54	−0.05	−0.03	−0.20	0.26
5 (*b*)	0.20	0.06	0	0.04	0	−0.20
		−0.48	−0.05	+0.01	−0.20	0.06
1 (*b*)	−0.5	0.50	0	−0.04	0	−0.10
		0.02	−0.05	−0.03	−0.20	−0.04
6 (*b*)	−0.25	−0.05	−0.11	+0.14	0.11	−0.07
		−0.03	−0.16	+0.11	−0.09	−0.11
2 (*b*)	−0.2	0	0.20	0	−0.03$_5$	0
		−0.03	0.04	0.11	−0.12$_5$	−0.11
7 (*b*)	0.2	0	−0.01	−0.10	0.10	0
		−0.03	0.03	0.01	−0.02$_5$	−0.11
5 (*b*)	−0.12	−0.04	0	−0.02	0	0.12
		−0.07	0.03	−0.01	−0.02$_5$	+0.01
1 (*b*)	−0.06	0.06	0	0	0	−0.02
		−0.01	0.03	−0.01	−0.02$_5$	−0.01
6 (*b*)	−0.04	0	−0.02	0.02	0.020	−0.01
		−0.01	0.01	0.01	0	−0.02

On simplifying assumptions which are customary, the same is true of rigidly-jointed structures. Here I would make full acknowledgment of Prof. Cross' priority in the root idea of continued whittling or "liquidation"; and I also admit freely that my own notions (though they seemed to come out of a clear sky) may have had origin in his work, with which I had some acquaintance before I started, but no real familiarity. My notions came while I was writing, for a book, an account of methods for calculating stresses in *braced* frameworks: it was for that reason that I felt concern from the first to establish convergence, and it is to this circumstance that I attribute the ease of our path to harder problems, some of which I shall describe. There came to me suddenly, quite sharp and clear, a mental picture of a way in which real structures might be loaded, *a way that could be followed in analysis*. If I can convey that picture, I may leave it to you (I think) to convince yourselves that I have already given its mathematical equivalent.

The ordinary "screw-jack" (*e.g.*, for automobiles) is a means whereby a controlled displacement may be

TABLE 2 (N.B. All forces have been multiplied by 10^6) (*Continued on next page*)

	Line	Imposed displacements	λ	F_1	F_2	F_3	F_4	F_5
Stage 1	1	Due to initial displacements (43)	*0.3*	−344,397	1,041,490	−650,659	383,151	−816,745
	2	Due to displacement corrections (46)		376,829	−501,980	672,426	−350,339	549,192
	3	Total forces		32,432	539,510	21,767	32,812	−267,553
	4	a_k (corrected)		1.006674	1.596358	0.957343	0.449137	−0.594696
	5	Products (3) × (4) ($\mathbf{F}_k.a_k$ and $T_k.a_k$)		32,648	861,251	20,839	14,737	159,113
	6	$(\mathbf{F}/T)_L = 0.08449$; $\sum_F/\sum_T = 0.075887$: hence, according to (21), *0.21551* $< \lambda <$ *0.224113*						
		$\Delta\lambda$ chosen for second stage = −0.077	*0.223*					
	7	Forces from line 3 (repeated)		32,432	539,510	21,767	32,812	−267,553
	8	$\Delta\lambda.T_k = -0.077T_k$		−77,514	−491,678	−58,972	−34,584	274,750
Stage 2	9	Corrected residuals (sums of lines 7 and 8)		−45,082	+47,832	−37,205	−1,772	7,197
	10	Displacement corrections (47)		37,595	−49,067	35,922	−279	−2,928
	11	Total forces		−7,487	−1,235	−1,283	−2,051	4,269
	12	a_k (corrected)		1.007154	1.627211	0.949575	0.440581	−0.602905
	13	Products (11) × (12) ($\mathbf{F}_k.a_k$ and $T_k.a_k$)		−7,541	−2,010	−1,218	−903	−2,574
	14	$(\mathbf{F}/T)_L = -0.000190$; $\sum_F/\sum_T = -0.000971$: hence, according to (21), *0.223190* $< \lambda <$ *0.223971*						
		$\Delta\lambda$ chosen for third stage = 0.0008	*0.2238*					
	15	Forces from line 11 (repeated)		−7,487	−1,235	−1,283	−2,051	4,269
	16	$\Delta\lambda.T_k = 0.0008T_k$		806	5,207	608	353	−2,894
Stage 3	17	Corrected residuals (sums of lines 15 and 16)		−6,681	3,972	−675	−1,698	1,375
	18	Displacement corrections (48)		6,495	−4,515	−150	1,388	−641
	19	Total forces		−186	−543	−825	−310	734
	20	a_k (corrected)		1.004014	1.627319	0.949130	0.439763	−0.602680
	22	Products (19) × (20) ($\mathbf{F}_k.a_k$ and $T_k.a_k$)		−186	−884	−783	−136	−442
		$(\mathbf{F}/T)_L = -0.0000835$; $\sum_F/\sum_T =$ −0.0001696; hence, according to (21), *0.2238835* $< \lambda <$ *0.2239696*						

imposed at some desired point: it is easy to imagine devices whereby this displacement can be recorded, together with the load sustained at any instant; also to visualize an arrangement in which every joint of a framework which would normally be free to move is provided with a jack of this kind to control its displacement. Suppose that initially the jacks fix the joints in positions such that the framework is not strained: then evidently, when the external loads are applied, they will be taken wholly by the jacks. Suppose that subsequently one jack is relaxed so that one joint is permitted to travel slowly through a specified distance: then load will be transferred from that jack to adjacent jacks and to the framework, and strain-energy will be stored in the latter. If the force on the jack which is relaxed had a component in the direction of the travel, that jack will be relieved of load, and strain-energy will be stored in the framework at the expense of the potential energy of the external forces.

Here then is an imagined process whereby a framework could be brought from the unstrained to the fully strained configuration; and as I said, every step is easy to follow in calculation. It is a physical, indeed an engineering, picture of workmen instructed to "ease away" in turn, each man coming into the limelight as his recorded load becomes the biggest.

All of this I have developed in a book published just a year ago, so I pass as quickly as possible to problems not yet written up (or at least, not in book form). But I would take leave for one exception, because without it I shall have difficulty in explaining more recent applications. That exception is vibrating systems—fairly described, I think, as a problem of aeronautical interest.

VIBRATING SYSTEMS

I need not explain to engineers the significance in relation to crankshaft vibrations of problems such as Fig. 3 illustrates: we need at times to know the frequencies, and the associated "modes of vibration," with which a given system can oscillate, left to itself. Nor shall I stop to elaborate "Rayleigh's principle," now widely known: it will merely give occasion for a

TABLE 2 (*Continued*)

F_0	Sums	T_1	T_2	T_3	T_4	T_5	T_6	Sums
537,645		1,000,000	5,137,929	912,157	499,041	−2,517,769	−1,196,407	
−576,392		6,674	1,247,503	−146,283	−49,904	−1,050,410	580,453	
−38,747		1,006,674	6,385,432	765,874	449,137	−3,568,178	−615,954	
−0.769943		1.006674	1.596358	0.957343	0.449137	−0.594696	−0.769943	
29,833		1,013,394	10,193,434	733,205	201,724	2,121,983	474,250	
	$(\sum_F = 1{,}118{,}422)$							$(\sum_T = 14{,}737{,}989)$
−38,747		1,006,674	6,385,432	765,874	449,137	−3,568,178	−615,954	
47,428								
8,681								
−8,043		480	123,413	−6,214	−8,556	−49,251	1,470	
638		1,007,154	6,508,845	759,660	440,581	−3,617,429	−614,484	
−0.768106		1.007154	1.627211	0.949575	0.440581	−0.602905	−0.768106	
−490		1,014,359	10,591,266	721,354	194,112	2,180,965	471,989	
	$(\sum_F = -14{,}736)$							$(\sum_T = 15{,}174{,}046)$
638		1,007,154	6,508,845	759,660	440,581	−3,617,429	−614,484	
−492								
146								
37		−3,140	429	−356	−818	1,351	184	
183		1,004,014	6,509,274	759,304	439,763	−3,616,078	−614,300	
−0.767875		1.004014	1.627319	0.949130	0.439763	−0.602680	−0.767875	
−140		1,008,045	10,592,663	720,678	193,391	2,179,336	471,705	
	$(\sum_F = -2{,}572)$							$(\sum_T = 15{,}165{,}819)$

formula, the only formula that I shall need this evening. Here it is:

$$V = p^2 T \quad (p = 2\pi n) \tag{1}$$

We say that in "normal vibration" all parts of a system move in phase, so as to come to rest, and reach their maximum velocities, simultaneously: p defines the frequency n, and then in Eq. (1) the left-hand side denotes the maximum potential energy (at the moment of rest), the right-hand side the maximum kinetic energy (at the moment of maximum velocity). Evidently these quantities are equal, since energy must be conserved: but now, suppose that we imagine constraints to make the mode what we assert, suppose that we *guess* the displacements. Then Rayleigh's principle, put briefly, tells us that we may be widely out in our guess about the mode, yet can use Eq. (1) to obtain a close estimate of the lowest natural frequency; *and that our answer will never come out too low.* Suppose then that by some other easy method we could arrive at a value known to be never too high: here would be knowledge almost if not quite as useful in practice as an "exact" result.

Early last year we devised this second rule; and we saw, moreover, how to bring these vibration problems within range of Relaxation Methods, with their power of continuous adjustment. The result is a tabular computation (Table 2), involving as before operations performed according to the judgment of the computer, but in which, now, as the residual forces are brought within ever-narrowing margins of uncertainty, the upper and lower limits imposed on p^2 continually approach, like pincers holding the wanted value always between their jaws. The pincers never close, but we can make their gap insignificant, as the table shows.

This is advance; but it would be advance of importance only to the academic, if only the lowest frequency were required; for Rayleigh gives that, with all the accuracy that normally has value. The practical gain is this, that *by our method we determine not only the lowest frequency but the corresponding mode;* for thereby we are enabled to start afresh, this time to determine the second lowest frequency. In fact, for a system of

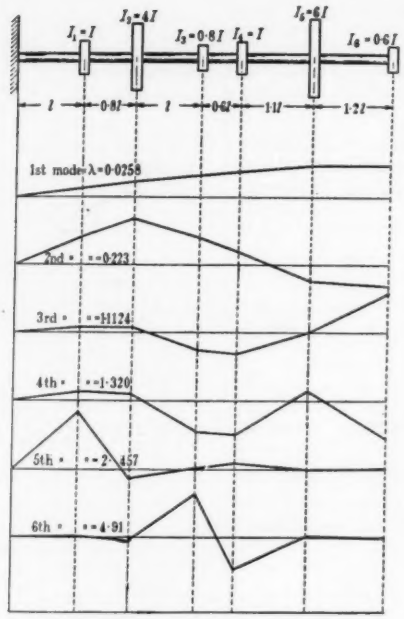


FIG. 3.

the kind that Fig. 3 exhibits, we can determine (and without excessive labor) *all* of the wanted modes and frequencies.

We have, moreover, found means of allowing for friction: we can, for example, deal with our former problem modified by the attachment of "vibration dampers" to two of the vibrating masses (Fig. 4). Table 3 gives (to those acquainted with these problems) the complete solution of this frictional problem.

CONTINUOUS SYSTEMS

Any method competent to deal with vibration problems can also, of course, be applied to problems of elastic stability. We have found that ours are in fact thus applicable, but results may not be published now. I can say, however, that in all classes of problem we have found it possible to deal not only with systems

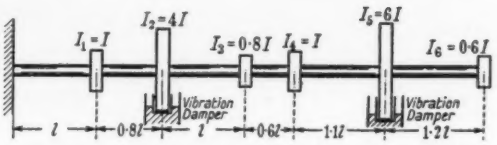


FIG. 4.

of finite freedom (such as I have described up to this point) but with continuous systems, such as are presented in Elasticity and in Hydrodynamics.

It is, in fact, in this field of two-dimensional problems that our work lies mainly now, and it is here that we are gaining our most encouraging results. We started in 1937, when we found it possible to solve (approximately) not only plane-potential problems of an ordinary kind (such as the torsion of non-circular bars—much studied of late by soap-film technique) but also problems which we had previously regarded as insoluble—cases of *plastic* torsion, *hollow* shafts of non-circular section, and the determination of magnetic lines in fields containing iron. Figs. 5–7 illustrate some published solutions of such problems.

Fig. 5 shows our solution, by a synthesis of two particular integrals, of the torsion problem for a pierced triangular shaft. The symmetry permits three diagrams to be combined in one: the component solutions are exhibited at the bottom, and the final solution at the top of the figure.

Fig. 6 exhibits the shear function for torsion of a triangular shaft as modified by the occurrence of "yield" at some limiting stress. Fig. 7 exhibits our calculation of magnetic lines in a field containing a triangular prism of iron, also (as a matter of historical interest) the solution obtained by Hele-Shaw and Hay (1900), with the aid of their very ingenious "streamline-technique." The correspondence will be seen to be very close.

I should emphasize that in every case the triangular boundary was taken merely for illustration. *What has been done in Figs. 5–7 could be done for any shape of boundary.*

TABLE 3

Complete Solution for Small Damping of the Torsional Problem of Fig. 6

$\lambda_1 = (-0.12583\,\mathbf{B} \pm i\,0.16081\,\sqrt{\mathbf{A}})$	$\lambda_2 = (-0.11126\,\mathbf{B} \pm i\,0.47325\,\sqrt{\mathbf{A}})$	$\lambda_3 = (-0.0090405\,\mathbf{B} \pm i\,1.05471\,\sqrt{\mathbf{A}})$
$a_1 = 1 \pm i\,0.08692\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 1.0040 \pm i\,0.00476\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 0.25832 \pm i\,0.07592\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_2 = 1.77927 \pm i\,0.12229\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = 1.62732 \mp i\,0.07687\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = 0.23515 \pm i\,0.06515\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_3 = 2.56899 \pm i\,0.15196\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 0.94913 \mp i\,0.02539\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = -0.83969 \mp i\,0.00809\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_4 = 3.01088 \pm i\,0.11793\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = 0.43976 \mp i\,0.03975\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = -1.03619 \mp i\,0.04004\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_5 = 3.73538 \mp i\,0.08188\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0.60268 \mp i\,0.10722\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0.12846 \mp i\,0.02790\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_6 = 3.83048 \pm i\,0.06864\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = -0.76787 \mp i\,0.23549\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = 1.89037 \mp i\,0.09874\,\mathbf{B}/\sqrt{\mathbf{A}}$
$\lambda_4 = (-0.031401\,\mathbf{B} \pm i\,1.14919\,\sqrt{\mathbf{A}})$	$\lambda_5 = (-0.01239\,\mathbf{B} \pm i\,1.56780\,\sqrt{\mathbf{A}})$	$\lambda_6 = (-0.0017448\,\mathbf{B} \pm i\,2.21649\,\sqrt{\mathbf{A}})$
$a_1 = 0.26235 \pm i\,0.04002\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 1.34688 \mp i\,0.02737\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 0.05797 \pm i\,0.00729\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_2 = 0.19510 \pm i\,0.01460\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = -0.22412 \mp i\,0.03730\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = -0.12364 \mp i\,0.01589\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_3 = -0.91920 \pm i\,0.07356\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 0.01565 \pm i\,0.00050\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 2.07896 \mp i\,0.00282\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_4 = -1.00507 \pm i\,0.09414\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = 0.14098 \pm i\,0.02231\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = -1.50198 \pm i\,0.00395\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_5 = 0.29761 \pm i\,0.07491\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0.01034 \mp i\,0.00404\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = +0.04964 \pm i\,0.00778\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_6 = -1.11143 \pm i\,0.00782\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = 0.00761 \pm i\,0.00276\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = -0.01336 \mp i\,0.00207\,\mathbf{B}/\sqrt{\mathbf{A}}$

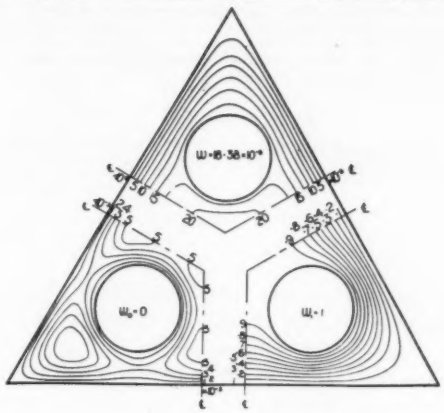


FIG. 5.

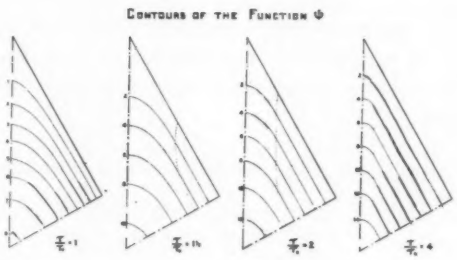


FIG. 6a.

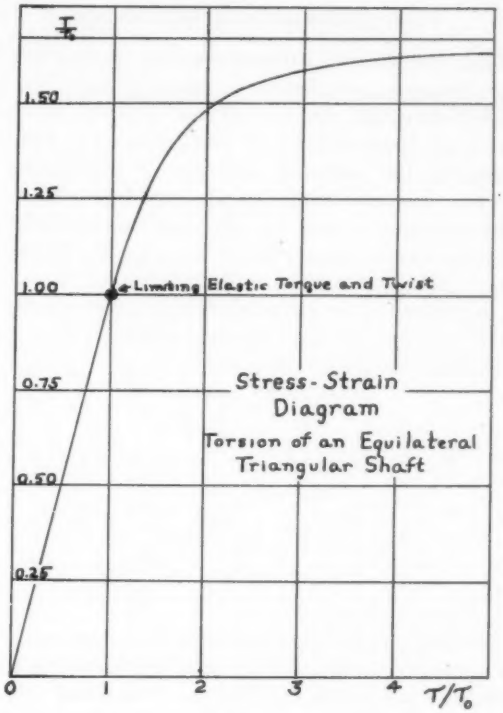


FIG. 6b.

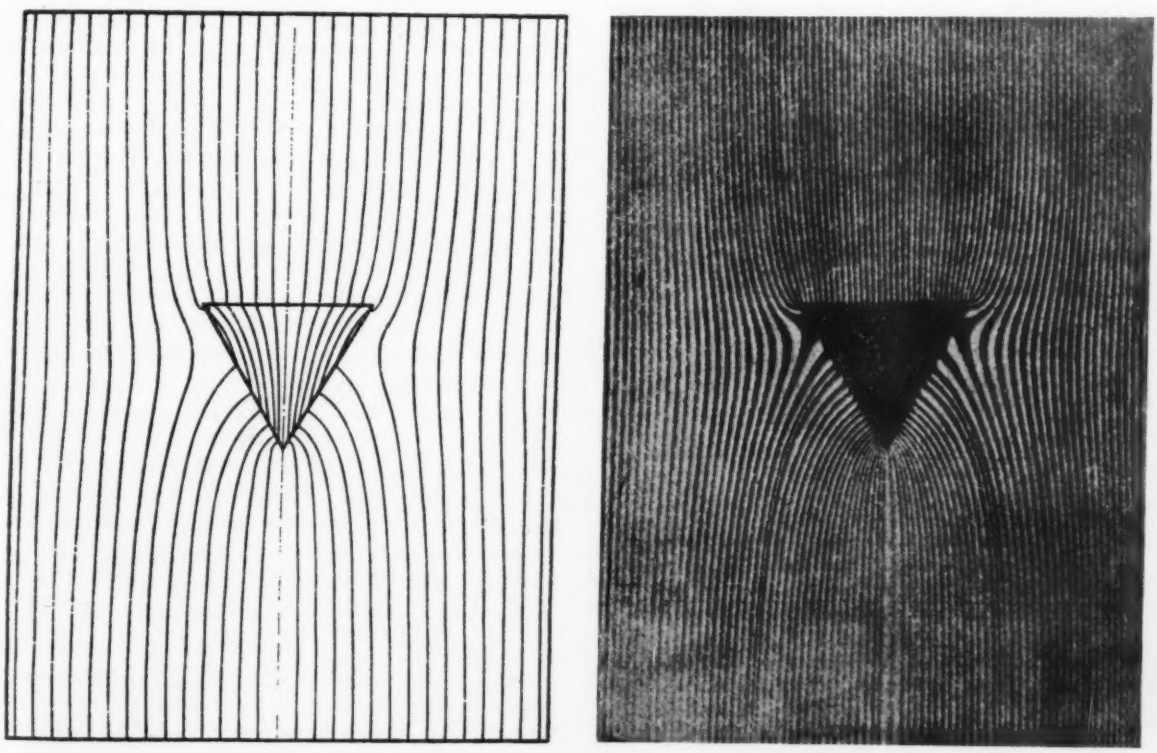


FIG. 7.

Finite-Difference Approximation

These are all early attempts, and to explain them I must continue my account of methods. First, for the exact differential equation of our problem we substitute an approximation in finite differences, thereby discarding the orthodox objective of determining a function *everywhere*, and setting ourselves instead to calculate its value at a large but finite number of chosen points, arranged symmetrically on a regular lattice or "net." For our purpose each nodal point or "knot" of the net must be surrounded by a fixed number of equal "strings" arranged symmetrically. There are 3 possibilities, and only 3 (Fig. 8).

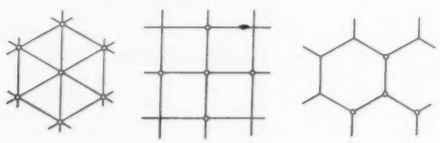


Fig. 8.

Finite-difference approximations have been employed by various investigators—notably Sheppard and Richardson in years preceding the first Great War. Without exception, so far as I know, they have (in effect) entailed square nets ($N = 4$). We have found that triangular nets ($N = 6$) entail no additional difficulties and in general yield more accurate results. Hexagonal nets ($N = 3$) do not appear to offer any advantage, but we have used the square and triangular indifferently, as each problem seemed to require. Sometimes use of both is convenient, *e.g.*, in our preparation (Fig. 9*) for the computation of flow of gas or steam through a convergent-divergent nozzle. Because we wanted to work on a square net, we were faced with the problem of conformally transforming the nozzle into a rectangle; but for that problem a triangular net was thought desirable on the score of accuracy.

We have published (1940) a detailed account of our methods as applied to the problem of conformal transformation. There are four such transformations which give promise of being useful: a diagram will explain them better than many words. (See Fig. 10.)

The Plane-Potential Problem

Though I have given examples from other fields, I need not labor, before this audience, the importance of the plane-potential problem to aeronautics. It is the first major problem in theoretical hydrodynamics, and must be solved if we would calculate stream-lines around an aerofoil or strut, must be solved *as a preliminary* if we would calculate the effects of compressibility. Experimental devices, to obviate the need of calculations, have been suggested frequently in recent years. It seems to have yielded unconditionally to computation based on relaxation methods, and my young men

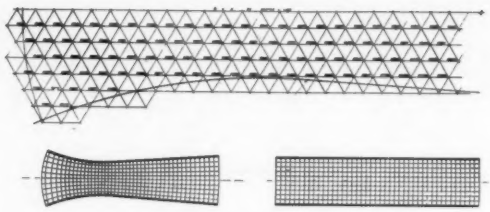


Fig. 9.*

take it for granted, now, that any solution they may need they can readily obtain for themselves, with all the accuracy that has value in physical applications. Lately we have been concerned to use our methods not as alternatives, but absolutely, for solving problems which by orthodox methods are intractable. I may not now give aeronautical examples, but perhaps it will suffice to give examples in the allied field of hydrodynamics. Here are three problems confronted in the theory of percolation—slow motion of fluid through the crevices of a porous material such as earth or sand. (Figs. 11–13.)

The Essence of the Relaxation Method

But what, in these examples, is the part of relaxation methods? I have told you only that we seek to compute the wanted function at points of a chosen "net"; I have not given details of the computations. Let us go back to first principles.

Speaking quite generally, our problem is to determine from its governing equations (in which I include boundary conditions) some wanted function z of two independent variables x and y. Here I would emphasize that when I speak of a function I do *not* imply that its mathematical form is known: thus from my standpoint height of a country above sea-level is a function of position east-and-west (the x-direction) and north-and-south (the y-direction), but only a very courageous man would try to represent it in any detail by a mathematical expression. We exhibit its variation in relief maps, or in contour maps which are a less elaborate alternative: similarly we *could* represent a mathematical expression, say $z = xy$, by a model in relief, but other means of representation exist. Here for example (Fig. 14) is a diagram from *Jahnke and Emde* which fulfils the same purpose exactly.

Now, a new point. I have talked of models constructed (or diagrams drawn) to represent known surfaces or functional relations; but it is quite as natural, when the model is simple, to imagine it *generated* by some physical means. And herein is seen the germ of Prandtl's well-known "membrane analogue": he first

* War conditions made it impossible to send from England the original drawings from which the diagrams of this lecture have been abstracted, and the figures in consequence have had to be prepared from the author's lantern slides. It is regretted that some have thereby lost definition of detail—notably Figs. 9–15.

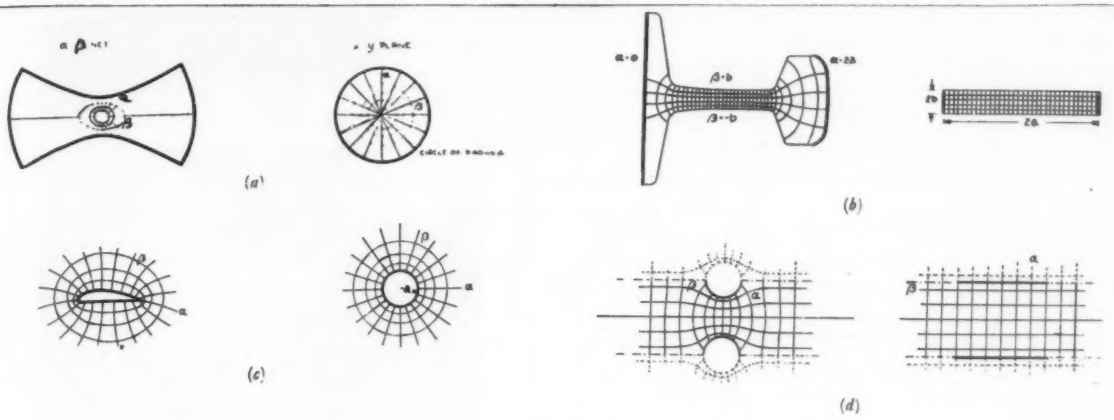


FIG. 10.

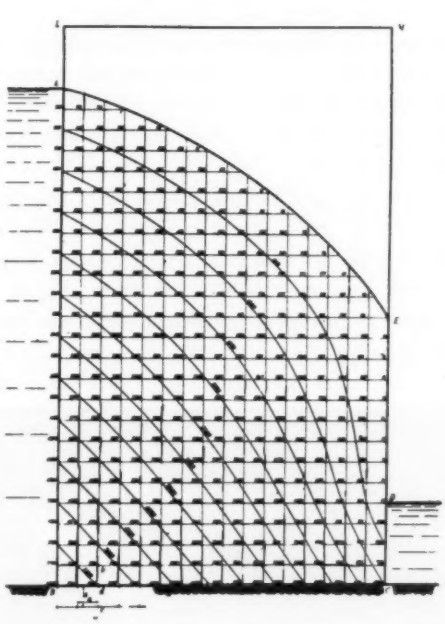


FIG. 11.

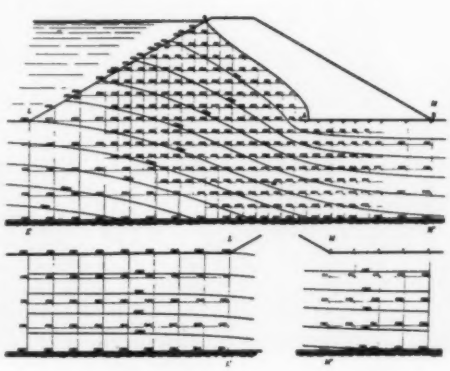


FIG. 12.

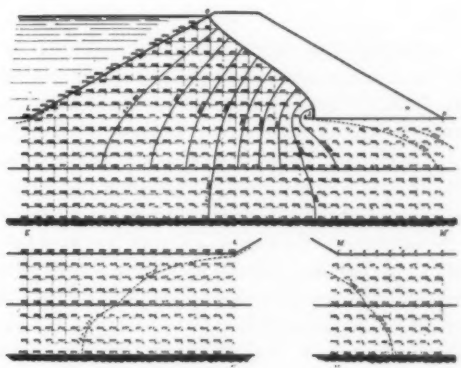
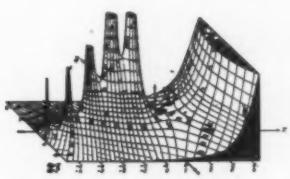


FIG. 13.



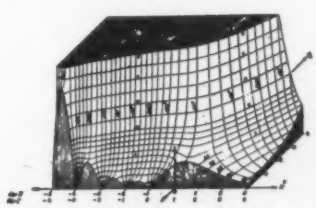


FIG. 14.

pointed out that a membrane like a soap-bubble, under appropriate conditions in regard to pressure and the like, does in fact solve a plane-potential problem and will (as Taylor and Griffith were quick to demonstrate) solve plane-potential problems for us if we choose.

I have said that in our computations we replace the

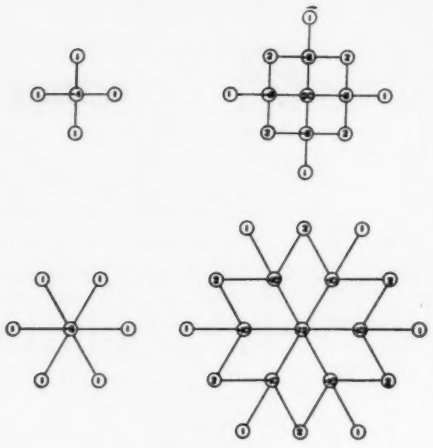


FIG. 15.

FIG. 16.

exact differential equation by a finite-difference approximation, then seek values at the nodal points of a regular "net": now, we see physical reality in this hitherto mathematical device. *The net of Fig. 8, pictured as strained by applied tensions, is our approximation to Prandtl's membrane;* it satisfies the finite-difference equation, and we can estimate the resulting error *by intuition*. So in the end our way is clear: for we can treat the net just as at starting I treated the framework; we can (in imagination) furnish it like the framework with adjustable constraints; we can impose displacements on the nodal points, in any order we may choose; *and throughout we may fix attention not on the wanted function* (*i.e., on displacements*) *but on loads not yet accounted for*, seeking to bring these within chosen margins of error. That is the essence of the Relaxation Method.

And in it lies, as I believe, the secret of our success: the use of relaxation patterns, typified in Fig. 15, to dispose of "forces" which are recorded no longer in tables, but now on reticular diagrams. Time alone limits the completeness of their liquidation and the fineness of our net.

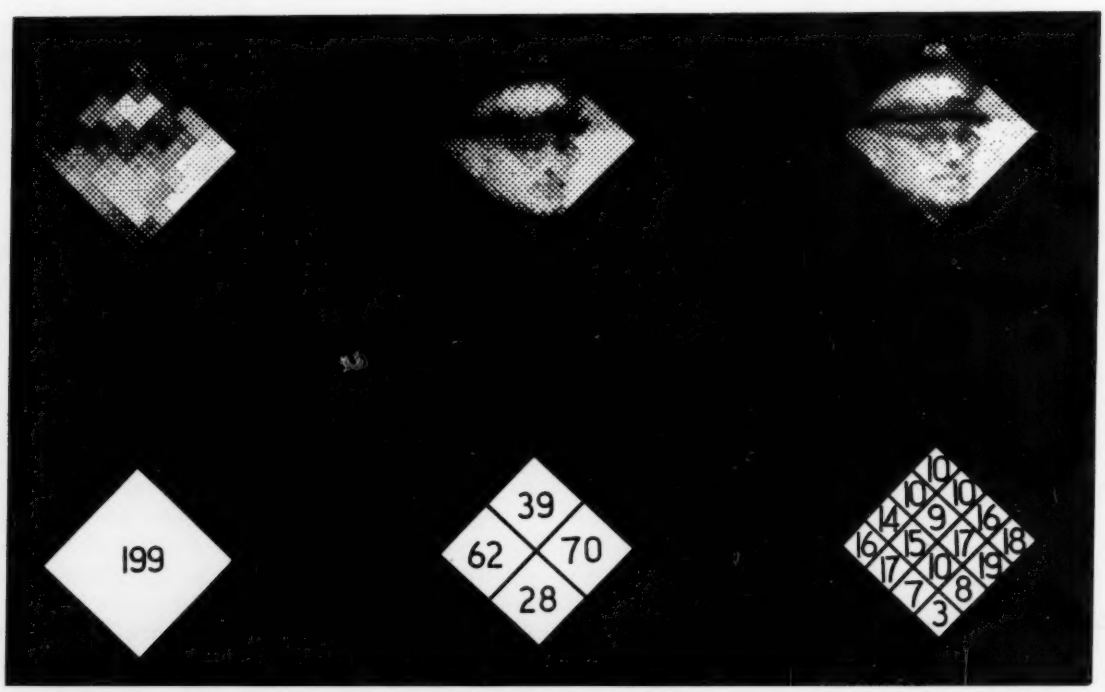


FIG. 17.

What have we at the finish? No formal solution, merely a "grid" of computed values such as Figs. 11–13 exemplify; coarse or fine as our effort has been short or sustained. This, let me emphasize, is what we must deduce from any formal solution before we can put it to practical use: the pure analyst, when having given us a formal solution he goes home to his tea, and from our standpoint, he has left his work half done! To any who object that it is not the whole picture, I answer that it is what we accept without question in a "half-tone" reproduction. Consider Fig. 16.*

A. N. Black, a member of my staff at Oxford, has pursued this line of thought somewhat further, using a microscope to analyze part of a half-tone reproduction taken from a newspaper. Each elemental square was viewed in turn, and an estimate made and recorded of the fraction of its area which was "black" (*i.e.*, inked): then, from the record of these estimates, a diagram was constructed on squared paper to have the same fractions inked of its corresponding squares. No particular convention was adopted—the inked fraction might have any shape or position in its square; but viewed from an appropriate distance the resulting diagram reproduces with surprising fidelity both the details and the general quality of the original photograph. Fig. 17 is, I think, self-explanatory.

Conclusion

I end, as I began, on a personal note. First, in this lecture I have thought it invidious to name any of my research assistants, when I could not mention all: do not conclude that the hardest work has been done by me! I owe much to our Ministries of Aircraft Production, of Home Security, and of Supply, also to our Department of Scientific and Industrial Research, that they have given me the means to retain such zealous collaborators.

Secondly, I could not even if I had wished have given this lecture a "war flavor," for the very good reason that wartime applications may not be divulged; and it may well seem to you, in consequence, that throughout the last two years we have been working in academic calm, seeking solutions for pure love of the quest, as is natural in times of peace. That is hardly a true picture of the facts, fortunate though we have been in the circumstances of our work at Oxford; for we have not let that work conflict with more immediate duties, and what I have given you tonight are but preliminaries and by-products of work which has throughout been focused on problems of the moment. What I have myself contributed (projects for trial next day) has been written for the most part on my knee in railway trains—the only place, it would seem, where leisure can be found in England today. But I admit to having tried to act on the belief that is in me, that the course of new research cannot be controlled—the researcher must follow where his problems seem to lead. Let him try, with all the will that he can muster, to set himself the goal of usefulness to the national effort; but let him *not* (if he would obtain results) try too precisely to foresee the road.

* For the photographs here reproduced as Figs. 16 and 17, the author is indebted to Mr. V. Belfield.

Discussion of the Lecture

At the conclusion of the lecture, Dr. Hunsaker called upon the following for their comment and discussion:

Dr. Hugh L. Dryden, National Bureau of Standards: "May I first offer my congratulations to the lecturer for the skill with which he has clothed the bare framework of a mathematical procedure of very general application, hidden its more forbidding aspects, and presented to us a physical structure with men hard at work with load-measuring devices and screw-jacks. The aeronautical world greatly appreciates Professor Southwell's contribution in reducing the solution of many of its most difficult theoretical problems to addition, subtraction, multiplication, and division, processes with which most of us are familiar and which we can perform with or without the aid of computing machines.

"In this country we have yet to exploit, to the same extent as elsewhere, the fruitful union of mathematics and engineering in which each discipline adjusts itself to the limitations and needs of the other. The pure mathematician is inclined to regard the union as a kind of miscegenation, while the practical engineer's relationship with mathematics is purely platonic. The greatest progress in aerodynamics, in structural theory, in meteorology, and in all other aspects of aeronautical science has come from mathematical theories guided by practical experimental results and from experiments interpreted in the light of mathematical theories. The empirical testing of a thousand specimens of an aircraft structure is worth less to the progress of aeronautics than one experiment on a single specimen carefully planned to determine which of the prevalent mathematical theories more nearly represents the facts. For mathematics is able to lead us, as the lecturer has stated, along new pathways into the field of the unknown and untried—which is more easily and safely explored on paper than in the air. It is hoped that Professor Southwell will carry his gospel of mathematics with a fringe to the mathematicians as well as to the engineers.

"I have no special qualification for discussing the more technical aspects which underlie the disarmingly readable account of the lecturer with its single mathematical formula in the text or to attempt to evaluate the relative convenience and merit of the relaxation method as compared with other methods of approximate numerical solutions of differential equations which are available. The lecturer has emphasized that it can be applied to problems for which the usual methods fail. Certainly one of the advantages of the relaxation method is that the bulk of the computation can be done by persons without special mathematical knowledge.

"Professor Southwell's book gives a good picture of the services of the relaxation method in the theory of structures and in the theory of vibrating systems. Those of us with special interest in aerodynamics look forward to the time when aerodynamic applications, for example, to the flow in boundary layers and to the flow of compressible fluids, which cannot now be disclosed, are made available to us.

"The members of the Institute are grateful to Professor Southwell for a most interesting lecture."

Dr. A. E. Lombard, Jr., Aircraft Branch, Office of Production Management: "Professor Southwell has given us a new approach to the general methods of successive approximation. Had it not been for the fact that he told us that this method was developed prior to the war, we would have suspected that he thought of it one night during a black-out in England. His method of relaxation sounds very much as though he had come out of a building on a dark night during a black-out and hit the first lamp post a rather violent bump whereupon he careened off, started down the street and hit the next lamp post a little less violently. Finally he has been able on this night to straighten out his course so that each successive lamp post down the street gets a lesser and lesser bump.

"The British scientists have always been famous for their development of these methods of successive approximation. Lord Rayleigh's methods for vibration analysis and Hardy Cross's methods for redundant structures are well known to us. Apparently, according to our lecturer tonight, the Southwell method is particularly suited in cases where there is an ample supply of bank clerks to carry out the work. The examples which he has shown us of the result of his method are stimulating and inspire us to investigate further the detailed workings of this particular method which shows definite suitability for the solving of certain problems.

"Our lecturer also has emphasized a most important point to be realized by aircraft engineers and scientists, namely, that we are working with approximate data and our attempts at accuracy should be consistent with the quality of the data. There is an ever increasing attempt in the aircraft industry to carry out calculations to a higher and higher degree of accuracy. This effort, when it is directed toward determining the fundamental physical constants with which we are working, is commendable, but when it is purely academic juggling of arithmetic it is a deterrent to progress and may impede the design and production of new aircraft."

Dr. John E. Younger, University of Maryland: "I am truly astonished and amazed at three features of Professor Southwell's presence among us: First, that he so nonchalantly makes the trip across the battle area of the Atlantic to honor us; second, that there is such close cooperation between his University and his government through him in this war-time emergency; and third, that he is succeeding in streamlining structural analyses for war-time speed. The second and third of these features we will do well to use as our pattern.

"If an engineer shows us how to build a new kind of airplane, a new kind of engine, or a new kind of wing flap, his contributions are always held with great acclaim because everyone can see the material object and the benefit it will be to the science of airplane design. But when an eminent scientist presents us with the nucleus of a new mathematics much better suited to our aeronautical structural problems than our orthodox methods and the resulting development is obviously an enormous stride in the perfection of airplane design and structural analysis, unfortunately no such popular acclaim may be expected. It is not spectacular, but it is the most needed improvement in the whole aircraft industry in this country. Structural design and structural analyses are lagging far behind other phases of the industry.

"There is too wide a gap between our mathematical elasticians and our structural engineers, so that the former, who are generally in charge of our research, are of little benefit to the latter, who are not concerned with ideal conditions and data but who must build out of available stock, which, as Prof. Southwell has said, has an *unavoidable uncertainty of physical data.*

"I have long felt that our methods of analysis of aircraft structures have been entirely inadequate for the purpose for which they were intended. They do not give the right answers. In general, the results of long mathematical analysis are no better for the practical structural analyses than an intelligent guess. We must bear in mind that even if an aeronautical structural engineer cannot solve the problem, he must nevertheless find an answer which will be satisfactorily accurate. If we can obtain a satisfactory result for a structural design, taking all the factors into consideration, by a series of elementary approximations, it appears to me that we have made great progress in stream-lining our design stress analysis processes.

"While we are on this subject, I wish to express the hope that Professor Southwell's methods will enable us to satisfactorily explore the regions where Hooke's law does not apply. Practically all our orthodox methods

are based upon Hooke's law, but practically all our aircraft structural analyses are out of its field. The orthodox method of analyzing a structure within a proportional limit and assuming that the results apply above the proportional limit is always depressing because we know that our final result is nothing but a poor guess which may in general have been made without any previous analysis.

"We attempt too often to give our research reports a tone of scientific glamour by a long mathematical analysis which quite successfully conceals the value of the research from most engineers, but is not justified by the nature of the research or the results. It appears to me that Professor Southwell's *new pathways* may quickly lead us to better coordination of our efforts."

Dr. George W. Lewis, N.A.C.A. (who was not present but who sent in the following comments): "The National Advisory Committee for Aeronautics extends Professor Southwell a very cordial welcome to the United States, and appreciates that in these disturbed times it was possible for him to visit the United States and present the Wright Brothers Lecture. Members of the Committee's staff are looking forward to the opportunity of having discussions with Professor Southwell, and it is hoped that his visit to the United States will be of sufficient length for him to have an opportunity to visit the Committee's laboratories. "I regret that it was impossible for me to attend the Wright Brothers Lecture, but I did have the opportunity of reading the lecture. It has been said that the work of Hardy Cross on moment distribution is the greatest contribution to structural engineering in the present generation. The 'new pathways' of Professor Southwell hold the promise of being to engineers generally what moment distribution has been to the structural engineers. The important thing to me is that the methods developed by Professor Southwell are not limited to aeronautics but can be applied in all branches of mathematical physics, of which engineering is one part. It is fortunate for us working in aeronautics to be among the first to have the benefits of this work. In the research of the National Advisory Committee for Aeronautics, relaxation methods have been used both in the solution of problems in the stability of structures and in the stress distribution of monocoque shells. The successful results obtained thus far indicate clearly that the use of this method will increase with time. It is regrettable that the present conditions do not permit Professor Southwell to give any details of his work on aeronautical subjects, which would, of course, be of particular interest to members of the Institute. Still, no matter how long we may have to wait for the release of this work, its intrinsic worth is such that the passage of time will not detract from its value."

Frank W. Caldwell, United Aircraft Corporation: "I take this opportunity to acknowledge the great debt which we owe to English science and particularly those branches of it that apply to aeronautics. Professor Southwell's lecture brings to our attention the research done by himself and his colleagues which is of such fundamental importance to the progress of the aeronautical sciences. I want to thank Professor Southwell personally and as President of the Institute of the Aeronautical Sciences for the valuable contribution which he has made in the form of this lecture."

Dr. Hunsaker then presented to Professor Southwell the Wright Brothers Lecture honorarium of $500 which is provided by the Vernon Lynch Fund. He thanked Professor George B. Pegram of Columbia University for again making the facilities of the University available for the lecture. With an expression to Professor Southwell of the appreciation of the Institute for his interesting and valuable lecture, the meeting was closed.

On Some Vortex Theorems of Hydrodynamics

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Catholic University of America

Dynamics of Incompressible Fluids

The motion of each element of an incompressible fluid takes place in coordination with that of all others in such a manner that no element of space becomes incompletely filled or overcrowded. The mathematical expression for this kinematic condition is the equation of continuity.

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

where $\mathbf{v}$ denotes the velocity of the fluid. This equation expresses the fact that the net outflow of fluid over any infinitely small closed surface, divided by the enclosed volume, is zero. This equation applies to every incompressible fluid either with or without viscosity, for steady and unsteady motions alike.

The mechanism enforcing this continuity of flow is a system of internal stresses between adjacent fluid elements. These stresses are called into being as soon as any influence comes into play tending to compress or rarefy the fluid. For non-viscous or ideal fluids this system of stresses consists simply of a distribution of normal stresses or pressures.

The basic problem of hydrodynamics is the determination of the fluid motion when the boundaries of the fluid and the external forces acting upon the fluid are given. In many applications the converse problem is encountered, namely the determination of a system of external forces which is necessary to bring about a given motion of the fluid. It is important to realize that the latter problem does not have a unique solution. There are an infinite number of possible force systems which result in the same fluid motion. In order to further limit the force systems it is necessary to impose additional restrictions regarding the nature of the force systems, such as, for instance, that the external forces act only through specified regions. The reason for this is that certain force systems merely tend to compress or to rarefy the fluid without tending to influence its motion. The action of gravity on a glass of water standing immovable is of that kind. Such external forces are counteracted by a modification of the pressure distribution; they are purely static and do not influence the fluid motion.

As a first step in an analysis of fluid motions such purely static force systems and their resultant static pressure distributions are dismissed from the investigation. Two force systems will be considered as equivalent if they have the same dynamic effect. Two equivalent force systems $\mathbf{f}_1$ and $\mathbf{f}_2$ will therefore differ by a purely static force system which can be balanced by a distribution of pressures P. This fact can be mathematically expressed as follows:

$$\mathbf{f}_2 - \mathbf{f}_1 = \nabla P$$

The negative gradient, called the anti-gradient of the pressure distribution P, represents the buoyancy force per unit volume incidental to the pressure distribution P and in this case balancing the static force $\mathbf{f}_2 - \mathbf{f}_1$.

The rotation of a gradient is always equal to zero, and thus

$$\nabla \times \nabla P = 0$$

Therefore

$$\nabla \times (\mathbf{f}_2 - \mathbf{f}_1) = 0$$

or

$$\nabla \times \mathbf{f}_2 = \nabla \times \mathbf{f}_1 \tag{2}$$

The condition for the equivalence of two force systems is therefore that they have equal rotation (curl) at all points. This statement holds also for viscous fluids because the static stress system of an immovable viscous fluid is likewise a pure pressure distribution.

This introduction is concluded by remarks on the external forces encountered in hydrodynamic problems. The gravity forces acting on a fluid of constant density form a static system and can of course be taken care of by a suitable pressure distribution in conjunction with suitable boundary conditions. With ordinary reference systems, there remain then as dynamic external forces acting on the fluid, those transmitted by foreign objects immersed within the fluid or forming the fluid container. If the objects are infinitely thin sheets or lamina, their effect can be entirely described by disregarding the lamina, considering the fluid as continuous, and considering only the forces transmitted by the lamina to the fluid. If the immersed objects are voluminous, or constitute the external fluid boundaries, they may likewise be disregarded, and the space occupied by them may be assumed to be filled with fluid at rest and of the same constant density as the

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* Lecturer in Aerodynamics.

fluid investigated. The reaction between the solids and the fluid is again described by considering the respective forces on the fluid only. In that way, every flow problem can always be reduced to an equivalent problem referring to the entire space being filled with the fluid in question and acted on by a system of external forces.

The Acceleration

Consider the motion of a perfect fluid (no viscosity). The motion may be steady or unsteady. Then the problem is to determine the force system, or more exactly, one of the many equivalent force systems which is required for the given fluid motion. According to Newton's Second Law the force $\mathbf{f}$ acting upon a small element of the fluid having unit volume is equal to the product of the mass ρ and the acceleration $\mathbf{a}$ of the element, that is $\mathbf{f} = \rho\,\mathbf{a}$. The mass ρ of the unit volume is called the mass density. The acceleration $\mathbf{a}$ may also be expressed as

$$\mathbf{a} = D\mathbf{v}/DT = (\partial\mathbf{v}/\partial T) + \mathbf{v}\cdot\nabla\,;\ \mathbf{v} \qquad (3)$$

(See reference 2, page 4.)
The second term in this expression may be transformed. From

$$\begin{aligned}\mathbf{v}\times(\nabla\times\mathbf{v}) &= \mathbf{v}\cdot\nabla\,;\ \mathbf{v} - \mathbf{v}\cdot\mathbf{v};\ \nabla \\ &= \mathbf{v}\cdot\nabla\,;\ \mathbf{v} - {}^1\!/_2\nabla\,\mathbf{v}\cdot\mathbf{v}\end{aligned}$$

it follows

$$\mathbf{a} = \partial\mathbf{v}/\partial T + {}^1\!/_2\nabla\mathbf{v}\cdot\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v}) \qquad (4)$$

From what has previously been discussed, the chief interest is in the rotation of the external forces, and hence in the rotation of the acceleration. This is equal to

$$\nabla\times\mathbf{a} = \partial\mathbf{u}/\partial T + \nabla\times(\mathbf{v}\times\mathbf{u})$$

where $\mathbf{u} = \nabla\times\mathbf{v}$. The term $\nabla\times\nabla\,(\mathbf{v}\cdot\mathbf{v})$ vanishes since the rotation of a gradient is zero.

For steady flows the term $\partial\mathbf{u}/\partial T$ vanishes by definition. The rotation of the acceleration then reduces to

$$\nabla\times\mathbf{a} = \nabla\times(\mathbf{v}\times\mathbf{u})$$

and thus

$$\nabla\times\mathbf{f} = \rho\,\nabla\times(\mathbf{v}\times\mathbf{u}) \qquad (5)$$

This leads at once to the basic theorem about the force distribution required to keep the motion of a perfect fluid steady.

Basic Vortex Theorem

A force distribution equal to the cross product of the velocity and of the vorticity, multiplied by the fluid density, maintains any motion of a perfect fluid as a steady motion.

This is not Kutta's theorem. The forces of airfoil theory are directly presented by the basic vortex theorem. The forces transmitted by an ideal propeller disc to the ideal fluid are not directly specified by the above theorem, but are merely equivalent to the specified forces.

In the absence of external forces, the force system of the Basic Vortex Theorem must vanish.

Steady Flow Criterion

The condition that the initial motion of a perfect fluid be steady without the action of external forces is that the cross product of the vorticity and of the velocity be irrotational.

This statement should be compared with the different but not inconsistent statement in reference 2, page 226.

Helmholtz' Vortex Theorem

An important progress of the theory was made by considering the rate of change of the circulation of the velocity along a closed path moving with the fluid, such that the path remains occupied by the same fluid particles. This integral is a Lagrangian quantity and has to be expressed in Eulerian terms. The integral considered is

$$D/DT\int d\mathbf{r}\cdot\mathbf{v}$$

This is equal to

$$D/DT\int d\mathbf{r}\cdot\mathbf{v} = \int(D\mathbf{d}\mathbf{r}/dT)\cdot\mathbf{v} + \int d\mathbf{r}\cdot(D\mathbf{v}/DT) \qquad (6)$$

the first right-hand term, being $\int d\mathbf{v}\cdot\mathbf{v} = {}^1\!/_2\mathbf{v}\cdot\mathbf{v}/$, is clearly zero, the integration being performed along a closed path and returning to the point from which it started, at which $\mathbf{v}\cdot\mathbf{v}$ must have the initial value. There remains therefore only the second right-hand term. It follows that the rate of change of the velocity circulation is equal to the circulation of the acceleration. That, multiplied by the density, gives the circulation of the forces. If these forces have no circulation, that is, if they are irrotational, the velocity circulation along a path moving with the fluid will not change.

Helmholtz' Theorem

In a perfect fluid, not under the action of external forces having rotation, the vortices move with the fluid.

Conversely, it requires such external forces to bring about fluid motions in which there is a relative velocity between the vortices and the fluid elements; and the larger this relative velocity is, the larger will be the forces required for equal vortex strength and density.

Generalization of the Basic Vortex Theorem

The Basic Vortex Theorem refers to steady motions only. Unsteady motions may now be considered, inquiring about a similar vortex theorem that may ease the burden of the mathematical investigator.

The most general unsteady flow cannot be brought about by external forces acting at regions of vorticity only, and thus being simply associated with the vor-

tices. For the terms $\partial \mathbf{v}/\partial T$ do not stand in any necessary or direct relation to the vortex system $\mathbf{u}$.

A direct relation between the forces and the vortices can, however, be pointed out for the special case that all vortices can be considered to move merely, with a velocity $\mathbf{w}$, without the creation or annihilation of any vortices. Each point of each vortex line is transferred through the distance $\mathbf{w}$, the new points are connected with each other to form the new vortex line, and the circulation or strength initially ascribed to the old vortex line is again ascribed to the new one. Let there be only one such isolated vortex line of concentrated vorticity. The forces can then easily be determined from the theory of potential flows. The strip composed of surface elements of width $\mathbf{w}$ and parallel to $\mathbf{u} \times \mathbf{w}$ extending along the vortex is then the only region where the potential of the difference of the initial and of the modified velocity distribution abruptly changes its value. This change is equal to the circulation of the vortex, $\mathbf{u}$. Hence a force $\rho(\mathbf{u} \times \mathbf{w})$ is necessary to bring about the change. The force for maintaining the steady flow was seen to be $\rho(\mathbf{u} \times \mathbf{v})$. Hence the total force $\rho \mathbf{u} \times (\mathbf{v} - \mathbf{w})$ results.

Generalized Vortex Theorem

The unsteady motions of perfect fluids require external forces equal to the cross product of the vorticity and its velocity relative to the fluid, multiplied by the density.

This very special demonstration is nevertheless conclusive of the general validity of the theorem.

Proceeding to a general proof of the generalized vortex theorem, it is known that the rotation of the force responsible for the unsteadiness has to be the space integral of $\rho \partial \mathbf{u}/\partial T$. If any vector is free of divergence, its space integral is equal to the surface integral of its flux multiplied by the radius vector, that is to say it is equal to the static moment of that flux. This follows from

$$\nabla \cdot \mathbf{u};\ \mathbf{r} - \nabla \cdot \mathbf{u}\,\mathbf{r} = \mathbf{u} \cdot \nabla;\ \mathbf{r} = \mathbf{u} \tag{7}$$

$$\int d\mathbf{o} \cdot \mathbf{u}\,\mathbf{r} = \int dS\,\mathbf{u}$$

(See also Eq. (28) of reference 3.) The space integral of a system of vectors free of divergence can therefore be expressed from the surface conditions of the region considered, or, otherwise expressed, vector modifications leaving the surface flux intact leave also the space integral intact.

Consider a region of the space and substitute for the vortex configuration prevailing after the motion $\mathbf{w}$ of the vortices another, modified one, namely the initial vortex configuration present before the motion $\mathbf{w}$ took place. To this system are superposed near the surface two incomplete vortex systems: (a) the vortex elements carried through the motion $\mathbf{w}$ from the outside into the inside of the region, and (b) vortex portions of original strength and direction, connecting the interior ends of the moved vortex portions with their original point of intersection with the surface.

This modified vortex system, free of divergence, gives the same integral as the vortex system actually prevailing after the motion $\mathbf{w}$. Compute the space integrals of the two incomplete vortex systems (a) and (b). As to (a), the motion $\mathbf{w}$ carries into the space element considered, through its surface element $d\mathbf{o}$, the vorticity $-(d\mathbf{o} \cdot \mathbf{w})\mathbf{u}$, the expression within the bracket being the flux of the velocity $\mathbf{w}$. The system (b), vortices of extension $\mathbf{w}$ and strength $d\mathbf{o} \cdot \mathbf{u}$, give elements $(d\mathbf{o} \cdot \mathbf{u})\ \mathbf{w}$. Hence, the total change of the vorticity will be the surface integral of

$$d\mathbf{o} \cdot \mathbf{u}\,\mathbf{w} - d\mathbf{o} \cdot \mathbf{w}\,\mathbf{u} = d\mathbf{o} \times (\mathbf{u} \times \mathbf{w}) \tag{8}$$

Reducing the space element considered to infinitely small size, then carrying out the surface integration (8) and dividing the outcome by the volume of the space element, leads to $\nabla \times (\mathbf{u} \times \mathbf{w})$ by the very definition of the rotation. This corresponds to a rotation of the unsteadiness force $\rho \nabla \times (\mathbf{u} \times \mathbf{w})$.

The rotation of the steadiness force was $\rho \nabla \times (\mathbf{v} \times \mathbf{u})$ according to the basic vortex theorem. Hence the rotation of the total force is seen to be equivalent to the rotation of the product of density times the cross product of vorticity and the relative vortex velocity.

Application of the Generalized Vortex Theorem

It has already been pointed out that the generalized vortex theorem relates to special flows only. It cannot be generally applied to each and every unsteady flow of a perfect fluid. The application is particularly called for if all points of each vortex line move with an equal relative velocity. This is always the case with two-dimensional unsteady flows. The theorem is then applicable. It is, for instance, applicable if all such vortices occupy a straight line and continue to occupy the same, the velocities at the points of the straight line being parallel thereto, and hence the forces being at right angles thereto. These conditions prevail in the investigations reported.[4]

In that case, the relative velocity of the vortices is as follows. Let x denote the Cartesian coordinate parallel to the line:

$$\partial \mathbf{u}/\partial T = \mathbf{w}(\partial \mathbf{u}/\partial x) \tag{9}$$

and hence

$$\mathbf{w} = (\partial \mathbf{u}/\partial T)/(\partial \mathbf{u}/\partial x) \tag{10}$$

Hence the force per unit length of the line, necessary to bring about the unsteady flow considered is equal to

$$\rho\{\mathbf{v} - [(\partial \mathbf{u}/\partial T)/(\partial \mathbf{u}/\partial x)] \times \mathbf{u}\} \tag{11}$$

By the integration of this force and of its moment, the total force and the total moment are directly obtained.

Further Generalization of the Basic Vortex Theorem

The limitation of the generalized vortex theorem to certain modes of fluid motion can be removed and,

formally at least, the theorem can be applied to the most general case. So far, it is necessary that there are vortices present in the first place, which by their relative motion may provide the force reaction. If vortices are absent, they may however be provided by the introduction of fictitious vorticity. This is a pair of systems of equally strong but opposite vorticity, superposed to each other and thereby canceling each other. The proposition is indeed academical, and probably devoid of technical usefulness. But it is highly instructive and of heuristic value. A larger completeness and coherence of the science and its geometric pictures is brought about. It can thus be demonstrated that the generalized theorem leads back to the basic Eq. (5).

Consider a space element dS at rest, bounded again by surface elements $d\mathbf{o}$. Let there act a force system $\mathbf{f}$ on the fluid occupying and surrounding the space element, said fluid moving there relative to the space element with the velocity $\mathbf{v}$. Introduce now such fictitious self-canceling double vorticity parallel to the surface elements and at right angles to the force. In accordance with the generalized vortex theorem, the two vortex systems are then moved across the surface elements, in opposite directions indeed, but also with opposite intensities, and therefore equally contributing to a transport of vorticity into the space element.

Before computing this, disregard those components of the forces already required for the maintaining of the steady motion, namely $\rho(\mathbf{v} \times \mathbf{u})$ according to the basic theorem. The force to be accounted for by the motion of the fictitious vortices is accordingly $\mathbf{f} - \rho(\mathbf{v} \times \mathbf{u})$. This force gives rise to a transport of vorticity through the surface elements

$$d\mathbf{u}/dT = \int d\mathbf{o} \times (\mathbf{f}/\rho - \mathbf{v} \times \mathbf{u})$$

Reducing the space element to small dimensions and unit volume and carrying out the integration gives finally

$$d\mathbf{u}/dT = \nabla \times \mathbf{f}/\rho - \nabla \times (\mathbf{v} \times \mathbf{u})$$

which corresponds exactly to Eq. (5), by substituting $\mathbf{f}/\rho$ for the acceleration.

Integral Theorems. The Momentum

The preceding theorems deal with the relation between the forces at each individual point and the vorticity at the same point. The following theorems are of an altogether different class, dealing only with the total or integrated value, one single numerical value or set of values, either of the momentum, or of the moment of momentum, or of the kinetic energy of a body of a perfect fluid. The substitution of the vorticity for the velocity itself reduces the computation work in all cases where the vorticity is restricted to surfaces or lines. Thus a space integration may be reduced to a surface or line integration.

First the momentum. The impulse applied to the fluid filling infinite space, on the occasion of bringing about a certain flow configuration, is not recognizable subsequently in the total momentum of the infinite body of fluid. During the creation of the motion, there occur in general impulse exchanges between portions of the fluid, disproportionally surpassing in magnitude the impulse applied externally, and rendering it unrecognizable. Portions of the fluid have acquired an infinite momentum in one direction, other portions an almost equal momentum in the opposite direction. The summing up process leads to an ambiguous result, because this result depends on the order in which the contributions of the various portions of the fluid are successively taken into account.

While thus a momentum of the infinite body of fluid is not uniquely ascertainable and insofar does not exist, it is nevertheless of practical value to have available one unique preferred value of the total momentum, for the computation of the impulse required for the creation of the motion, and for some computations related thereto. A method how to define and obtain such preferred value is proposed and explained in reference 3. The momentum is computed from the fictitious irrotational motion of similar fluid imagined to be inclosed within the surface of the immersed solid, which motion agrees with the actual outside irrotational motion at all points of the surface. Of course, for a finite body of fluid there is no inherent difficulty in determining the momentum. There is, however, a particularity of the fictitious flow not found in physically real flows. In spite of the assumed constant density, and not inconsistent therewith, the flow may possess divergence; the fluid must be supposed to appear or disappear at certain regions.

The existing theorems serving for the computation of the momentum and the related quantities in case of a finite fluid body are therefore in need of a revision and particularly exact statement, to establish whether they also hold for the case of divergence. The variety of theorems will at the same time be increased by a few similar theorems which are possibly new and relate to the moment of momentum. At present, however, it will be necessary to dispose of the momentum itself.

The momentum of the fluid contained within a certain region of space has been expressed in reference 3, Eq. (28) by means of its divergence or source and sink system including its boundary flux system. It is there shown that the total momentum is equal to the static moment of the divergence and of the boundary flux multiplied by the density.

This integral gives the total momentum regardless of the presence of or coexistence of any vortex system, including the "roll," the surface integral of the cross product of the surface elements and the velocity. That is to say, the momentum is expressed in terms of the divergence only, for the boundary flux is equivalent to and complimentary to the divergence, becoming genuine divergence as soon as the space beyond the boundary

is supposed to be filled with resting fluid and included in the integration.

The static moment of the divergence is the prototype for other expressions containing the rotation and valid nevertheless for any coexisting divergence distribution. The disregard of either the rotation or the divergence in either case is only formal, and the disregarded quantity not explicitly contained in the integral expression still influences the final result. For it is still associated with the boundary conditions, and so makes itself felt in the boundary term if the truth requires that. If the fluid fills the entire space, the external regions are supposed to be occupied by resting fluid, and this presupposes and requires certain mutual relations between the divergence and the rotation or its absence by virtue of which certain integrals of one are also determined by the other.

The entire class of integral theorems now under discussion have this in common: that their surface terms stand in a unique relation to their space terms, in that they are fundamentally the same specially transformed for surface effects. The divergence is replaced by the flux, and the rotation is replaced by the roll. The surface terms appear from the space terms by partial integration or application of Gauss' theorem, which constitutes a generalization of Green's theorem.

The possibility of these theorems is intimately connected with the fact that ∇ designates a differentiation dimensionally equivalent to a division by a length. This division effect can be counterbalanced by a suitable multiplication by $\mathbf{r}$, the radius vector.

All this is most clearly exhibited in the divergence integral for the momentum. The space term $(\nabla \cdot \mathfrak{v})\mathbf{r}$, is dimensionally correct, being a vector; and one factor $\mathbf{r}$ counterbalancing the differentiation ∇. Only $\mathfrak{v}$ is differentiated, $\mathbf{r}$ is not. The surface term $d\mathbf{o} \cdot \mathfrak{v}\,\mathbf{r}$ results from the space term by replacing ∇ by $d\mathbf{o}$, or the divergence by the flux. The significant part of the moment, the space integral over $\mathfrak{v}$, results from the above space term by dislodging ∇ from $\mathfrak{v}$ and connecting it in direct association with $\mathbf{r}$, preserving otherwise the vector-analytical equivalence.

The same sequence of steps may now be applied to the expression

$$(\nabla \times \mathfrak{v}) \times \mathbf{r}$$

which is a vector of equal geometric dimension as

$$(\nabla \cdot \mathfrak{v})\,\mathbf{r}$$

The complemental surface term is $(d\mathbf{o} \times \mathfrak{v}) \times \mathbf{r}$, which can be written down without any analytical effort. The turning over of the differentiation ∇ from $\mathfrak{v}$ to $\mathbf{r}$ is a little more complicated than before, as long as one is restricted to ordinary dyadics. It is then necessary to dissolve the expression first to make all cross products disappear.

$$(\nabla \times \mathfrak{v}) \times \mathbf{r} = \nabla\,;\ \mathfrak{v} \cdot \mathbf{r} - \mathfrak{v};\ \nabla \cdot \mathbf{r}$$

Its complement:

$$\nabla\,;\ \mathbf{r} \cdot \mathfrak{v} - \mathfrak{v} \nabla \cdot \mathbf{r} = -2\,\mathfrak{v}$$

Hence

$$\int dS(\nabla \times \mathfrak{v}) \times \mathbf{r} - 2\int dS\,\mathfrak{v} = \int (d\mathbf{o} \times \mathfrak{v}) \times \mathbf{r} \quad (12)$$

$$\int dS\,\mathfrak{v} = {}^1/_2 \int dS(\nabla \times \mathfrak{v}) \times \mathbf{r} - {}^1/_2 \int (d\mathbf{o} \times \mathfrak{v}) \times \mathbf{r} \quad (13)$$

Momentum Theorem

The momentum of the fluid occupying a region of space is equal to the sum of half the vector moments of its vorticity and of its surface roll, multiplied by the density.

The factor ${}^1/_2$ can easily be verified for the case of a very small cubical space element being occupied by fluid flowing parallel to one edge. The space integral becomes then zero. The surface integral, without the factor ${}^1/_2$, gives then twice the volume multiplied by the velocity.

The theorem is also set forth in reference 2, Section 152, in a different form, not inconsistent with the one just given. The factor ${}^1/_2$ is there also correctly stated.

The Moment of Momentum

No suitable expression containing the divergence only can be found for the expression of the moment of momentum. It is indeed apparent from elementary considerations that the same divergence may be associated with fluid motions having different moments of momentum.

Three expressions present themselves as a first choice for obtaining integral expressions for the moment of momentum containing the rotation only. These are

(a) $\nabla \times \mathfrak{v}\,\mathbf{r} \cdot \mathbf{r}$, the moment of inertia of the rotation

(b) $((\nabla \times \mathfrak{v}) \times \mathbf{r}) \times \mathbf{r}$ the vectorial moment of inertia of the rotation, and

(c) $$((\nabla \times \mathfrak{v}) \times \mathbf{r}) \times \mathbf{r} - (\nabla \times \mathfrak{v}) \frac{(\mathbf{r} \cdot \nabla \times \mathfrak{v})^2}{\nabla \times \mathfrak{v} \cdot \nabla \times \mathfrak{v}}$$

an apparently more complicate expression, but adaptable to easy expression in Cartesian coordinates.

Case (a)

$$\int dS\,\nabla \times \mathfrak{v}\,\mathbf{r} \cdot \mathbf{r} = \int d\mathbf{o} \times \mathfrak{v}\,\mathbf{r} \cdot \mathbf{r} + \int dS\,\mathfrak{v} \times (\nabla \mathbf{r} \cdot \mathbf{r})$$

$$= \int d\mathbf{o} \times \mathfrak{v}\,\mathbf{r} \cdot \mathbf{r} + 2\int dS\,\mathfrak{v} \times \mathbf{r} \quad (14)$$

Hence

$$\int dS\,\mathfrak{v} \times \mathbf{r} = {}^1/_2 \int dS(\nabla \times \mathfrak{v})\,\mathbf{r} \cdot \mathbf{r} - {}^1/_2 \int d\mathbf{o} \times \mathfrak{v}\,\mathbf{r} \cdot \mathbf{r} \quad (15)$$

First Moment of Momentum Theorem

The moment of momentum of a body of fluid is equal to one-half the sum of the common moment of inertia of its vorticity and of the same moment of momentum of its surface roll, multiplied by the constant density.

Case (b)

$$((\nabla \times \mathfrak{v}) \times \mathbf{r}) \times \mathbf{r} = (\nabla \times \mathfrak{v})\,\mathbf{r} \cdot \mathbf{r} - \nabla \times \mathfrak{v} \cdot \mathbf{r}\,\mathbf{r} \quad (16)$$

$$\int dS(\nabla \times \mathfrak{v})\,\mathfrak{r}\cdot\mathfrak{r} = \int d\mathfrak{o} \times \mathfrak{v}\,\mathfrak{r}\cdot\mathfrak{r} + 2\int dS\mathfrak{v} \times \mathfrak{r} - \int dS(\nabla \times \mathfrak{v})\cdot\mathfrak{r}\,\mathfrak{r} \quad (17)$$

$$= -\int dS\mathfrak{r} \times \mathfrak{v}\cdot\nabla\,;\mathfrak{r} - \int d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r}\,\mathfrak{r} - \int dS\mathfrak{v}\cdot\nabla \times \mathfrak{r}\,\mathfrak{r}$$

$$= \int dS\,\mathfrak{v} \times \mathfrak{r} - \int d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r}\,\mathfrak{r} - \mathfrak{o} \quad (18)$$

whence

$$\int dS\,\mathfrak{v} \times \mathfrak{r} = -\int dS(\nabla \times \mathfrak{v})\cdot\mathfrak{r};\,\mathfrak{r} + \int d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r};\,\mathfrak{r} \quad (19)$$

which expresses the moment of momentum in terms of the dyadic moment of inertia of rotation and roll. Combining (15), (17) and (18)

$$\int dS\,\mathfrak{v} \times \mathfrak{r} = {}^1/_3\int dS((\nabla \times \mathfrak{v}) \times \mathfrak{r}) \times \mathfrak{r} + {}^1/_3\int ((d\mathfrak{o} \times \mathfrak{v}) \times \mathfrak{r}) \times \mathfrak{r} \quad (20)$$

Second Moment of Momentum Theorem

The moment of momentum of a body of fluid is equal to one-third of the sum of the vectorial moment of inertia of its vorticity, and of the same moment of inertia of its surface roll, multiplied by the constant density.

Case (c)

The first term is the same as under (a). The second term,

$$\nabla \times \mathfrak{v}\,\frac{(\mathfrak{r}\cdot\nabla \times \mathfrak{v})^2}{\nabla \times \mathfrak{v}\cdot\nabla \times \mathfrak{v}}$$

leads to a complement containing as a factor $(\mathfrak{r} \times \nabla\cdot\mathfrak{v})$ and is accordingly zero. Hence the factor ${}^1/_2$ appears again.

This third theorem is stated in a different form, but consistent with the form here given, in reference 2, Section 152. A numerical example for its application is there also given. No references are known to the author for the first two or three moment of momentum theorems.

The Kinetic Energy

Flows free of divergence and flux, as well as flows free of rotation and roll, possess each a definite kinetic energy. It becomes therefore apparent that the expression for the integral must contain both divergence and rotation. It must furthermore be of second order in either derivative and the velocity.

It is sufficient for the application that each term contain the rotation or divergence one time as factor only, because this is enough to restrict the integration to limited regions if that can be done at all.

An expression to start from, which leads to the aim is as follows:

$$\nabla \times \mathfrak{v}\cdot\mathfrak{r} \times \mathfrak{v} + \nabla\cdot\mathfrak{v}\,\mathfrak{r}\cdot\mathfrak{v} = \quad (21)$$

$$(\nabla\,;\mathfrak{v}\cdot\mathfrak{r})\cdot\mathfrak{v} - {}^1/_2\nabla(\mathfrak{v}\cdot\mathfrak{v})\cdot\mathfrak{r} + (\nabla\cdot\mathfrak{v})\mathfrak{r}\cdot\mathfrak{v} = \quad (22)$$

$$\nabla\cdot(\mathfrak{r}\cdot\mathfrak{v}\,\mathfrak{v}) - {}^1/_2\nabla((\mathfrak{v}\cdot\mathfrak{v})\cdot\mathfrak{r}) - \mathfrak{v}\cdot\nabla\,;\,\mathfrak{r}\cdot\mathfrak{v} + {}^1/_2(\nabla\cdot\mathfrak{r})\mathfrak{v}\cdot\mathfrak{v} \quad (23)$$

$$\int dS[\nabla \times \mathfrak{v}\cdot\mathfrak{r} \times \mathfrak{v} + \nabla\cdot\mathfrak{v}\mathfrak{r}\cdot\mathfrak{v}] + {}^1/_2\int [d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r} \times \mathfrak{v} + d\mathfrak{o}\cdot\mathfrak{v}\,\mathfrak{r}\cdot\mathfrak{v}] = {}^1/_2\int dS\,\mathfrak{v}\cdot\mathfrak{v} \quad (24)$$

since

$$d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r} \times \mathfrak{v} = \mathfrak{v} \times d\mathfrak{o}\cdot\mathfrak{v} \times \mathfrak{r} = \mathfrak{v}\cdot d\mathfrak{o} \times (\mathfrak{v} \times \mathfrak{r}) = \mathfrak{v}\cdot\mathfrak{r}\,d\mathfrak{o}\cdot\mathfrak{v} - \mathfrak{v}\cdot\mathfrak{v}\,d\mathfrak{o}\cdot\mathfrak{r} \quad (25)$$

$$\therefore -{}^1/_2\,d\mathfrak{o}\cdot\mathfrak{r}\,\mathfrak{v}\cdot\mathfrak{v} = {}^1/_2\,d\mathfrak{o} \times \mathfrak{v}\cdot\mathfrak{r} \times \mathfrak{v} - {}^1/_2\,d\mathfrak{o}\cdot\mathfrak{v}\,\mathfrak{r}\cdot\mathfrak{v} \quad (26)$$

Energy Theorem

The kinetic energy of a body of fluid is equal to the sum of the space integral of the dot product of the vector moment of the velocity and the rotation, of the space integral of the product of the scalar moment of the velocity and the divergence, and of one-half the corresponding surface expressions.

The factor ${}^1/_2$ in the surface terms is the consequence of the quadratic order of the terms. When a boundary is backed up by resting fluid and the integration extended beyond it, the surface terms becoming space terms, the average velocity at the points of the former boundry is only one-half of the velocity at one side of the boundary, being the arithmetic means between that velocity and zero. Hence the same transition as before requires now an additional factor ${}^1/_2$, and the same is automatically obtained by the application of vector analysis.

A theorem partially equivalent to the last one is discussed in reference 2, Section 153.

The Kutta-Joukowski Theorem

The so-called Kutta-Joukowski theorem about the lift of a wing system moving with constant velocity in fluid otherwise at rest is another integral theorem, in that it gives only the total lift but not the distribution of the lift. The integrand differs from the terms prescribed by the basic vortex theorem in that the vorticity is multiplied by the constant velocity of advance, but not by the local flow velocity.

This theorem can be generalized to apply to more general steady fluid motions, having vorticity. The entire space is divided into a specified region, for which the total force is to be computed, and into the remaining exterior space. Fictitious velocities within the specified region are now computed by means of Biot Savart's rule from the vortex elements of the exterior region only. These velocities are substituted for the actual local velocities in the expression for the forces as specified by the basic vortex theorem. All these forces are then added up. The error committed by the use of the fictitious forces instead of the true local forces cancels out in the sum or integral, as will readily appear from the examination of Biot Savart's rule, considering that in the force integral all pairs of vortex elements appear twice as products with opposite sign. The correct total of the moments will be similarly obtained.

In Kutta's case, these fictitious velocities are the constant velocity of advance.

Generalized Kutta Theorem

The resultant of all forces necessary to act on a perfect fluid within a specified region to keep the motion steady can be computed by combining the velocities within that region, as determined from the exterior portion of the vortex system only, with the vortex system within the region, using otherwise the basic vortex theorem.

Viscosity Forces

Concluding the discussion of some vortex theorems, it may be pointed out that the force per unit volume of the incompressible fluid ascribable to viscosity can also be simply expressed by means of the vorticity. This force is stated in most textbooks to be

$$\mathbf{f} = \mu \nabla \cdot \nabla \mathbf{v} \tag{27}$$

that is, it is equal to the Laplacian of the velocity, multiplied by the viscosity modulus. See reference 2, page 547, Eq. (4); reference 1, Vol. 3, page 49, Eq. (6.5).

To transform this expression (27) into one containing the vorticity, there is combined the equation

$$\nabla \cdot \nabla \mathbf{v} - \nabla \nabla \cdot \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) \tag{28}$$

with

$$\nabla \cdot \mathbf{v} = 0 \tag{28a}$$

The viscosity force results then as

$$\mathbf{f} = \mu \nabla \times (\nabla \times \mathbf{v}) = \mu \nabla \times \mathbf{u} \tag{29}$$

Viscosity Force Theorem

The resultant force per unit volume ascribable to the viscosity is equal to the rotation of the vorticity, multiplied by the viscosity modulus of the non-compressible fluid.

It follows directly that this resultant force must vanish within the fluid in case of a potential flow.

The modification of the *Steady Flow Criterion* by the inclusion of this viscosity term is easily suggested.

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A Suggestion for Columns of Varying Section

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SUMMARY

The analysis of a slender column of varying section is usually difficult. This paper suggests that the problem of the varying column be approached from the point of view of determining the variation of section which will give a certain previously designated property. Three such properties are mentioned:

(1) Constant M/EI along the length, under the critical load.
(2) Constant Mc/EI under the critical load.
(3) Minimum use of material.

Cases (1) and (2) are represented by examples. Cases (1) and (2) come near to achieving (3). Case (1) is quite easy to apply.

INTRODUCTION

THE TERM COLUMN as used herein refers to a slender compression member in which failure will occur by buckling. In the analysis of such a member, it is supposed that it is subjected to its critical load, that is, the load which will cause it to bend, but which it can support when in the bent condition. Under any smaller load, the member will remain straight.

The logical reason for using a column of varying section, instead of a prismatic column, is to obtain certain properties not possessed by the latter. This being so, it appears that the most efficient method of dealing with a column of varying section is to decide what properties are desirable, and then to determine the variation of section which gives them; in many cases, this presents an easier problem than the analysis of a column of arbitrarily varying section. Properties which might be considered desirable in a column are:

(1) Constant M/EI along the length, under the critical load. M, E and I represent bending moment, modulus of elasticity and moment of inertia of cross-section, respectively. This is an easy criterion to apply. Also a buckling failure, for the type of member under consideration, is a bending failure. If M/EI is constant, the bending, as represented by the curvature, is the same at all points.

(2) Constant Mc/EI under the critical load. Here c is the distance from the neutral axis of the cross-section to its extreme fiber. If this condition is met, the maximum stress will be the same at all cross-sections.

(3) Minimum use of material. The first two conditions present comparatively easy problems. The third requires the use of Calculus of Variations. Blasius† has solved this problem for a few simple cases. In some of them, the results are the same as those obtained by trying for properties (1) or (2).

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* Assistant Professor of Mechanics.

† Blasius, H., *Trager kleinster Durchbiegung und Stäbe grösster Knickfestigkeit bei gegebem Materialverbrauch*, Zeitschrift für Mathematik und Physik, Vol. 62, page 182, 1913–14.

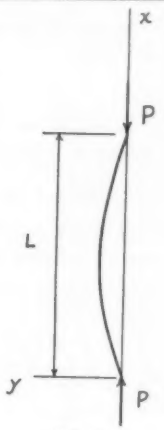

FIG. 1.

CENTRALLY LOADED COLUMN OF CONSTANT M/EI UNDER CRITICAL LOAD P. NO END RESTRAINT

In Fig. 1, consider first the centrally loaded column which is simply supported at the ends. For this case, $M = -Py$. It is customary in the analysis of slender columns to assume that the bending is similar to that of a beam, to the extent that the same differential equation may be used, namely, $d^2y/dx^2 = M/EI$. Then

$$d^2y/dx^2 = -Py/EI$$

Putting

$$Py/EI = A \qquad (1)$$

a constant, $d^2y/dx^2 = -A$, which integrates to $y = (-Ax^2/2) + Bx + C$. B and C are constants of integration. From the condition that $y = O$ when $x = O$, $C = O$. Using $y = O$ when $x = L$

$$O = (-AL^2/2) + BL;\ B = AL/2$$

Then

$$y = (A/2)\,(Lx - x^2)$$

Eq. (1) can be rewritten as, $y = AEI/P$. Equating the two values for y, $AEI/P = (A/2)\,(Lx - x^2)$.

$$I = (P/2E)(Lx - x^2) \qquad (2)$$

This gives I as a function of x. The maximum I is required at the center and is, $I_m = PL^2/8E$. No restric-

tions on the shape of the cross-section have been imposed in the above. The only requirement is that the bending axes of all sections be parallel.

Blasius investigated the problem of the column of rectangular section of constant depth and varying width. He found that Eq. (2) gives the column of minimum volume for such a case, on the assumption that the bending is in one plane parallel to the depth. This assumption would not appear to be well founded, since the width is zero at the ends.

Centrally Loaded Column of Constant Mc/EI Under Critical Load P. No End Restraint

In this case, it is necessary to take the shape of cross-section into account. The case considered here is that in which the section remains of the same shape, circular or square, for instance, while changing in size. With this limitation, $I = mc^4$, where m is a constant which depends on the shape; for a circular section, $m = \pi/4$; for a square section, $m = 4/3$. From Fig. 1,

$$d^2y/dx^2 = M/EI = -Py/EI$$

Thus, $Mc/EI = -Pyc/EI$. Putting $Pyc/EI = A$, a constant, and using $I = mc^4$, $Py/Emc^3 = A$

$$c = (Py/EmA)^{1/3} \tag{3}$$

Also

$$Py/EI = A/c$$

and substituting the value of c from Eq. (3)

$$Py/EI = (A^3mE/Py)^{1/3} = K^2/y^{1/3} \tag{4}$$

where

$$K^2 = (A^4mE/P)^{1/3}, \text{ a constant.}$$

Then

$$d^2y/dx^2 = -Py/EI = -K^2/y^{1/3}$$

This can be integrated by use of the integrating factor, $2(dy/dx)$.
Thus,

$$2(dy/dx)(d^2y/dx^2) = -2(K^2/y^{1/3})(dy/dx)$$

After integration

$$(dy/dx)^2 = 3K^2(d^2 - y^{2/3}) \tag{5}$$

in which the constant of integration appears as $3K^2d^2$. A change of variables, letting $y = z^3$, leaves Eq. (5) as,

$$z^2dz\sqrt{d^2 - z^2} = (K/\sqrt{3})dx$$

This integrates to

$$-(z/2)\sqrt{d^2 - z^2} + (d^2/2)\sin^{-1}(z/d) = (Kx/\sqrt{3}) + C_1$$

Changing back to y,

$$-(y^{1/3}/2)\sqrt{d^2 - y^{2/3}} + (d^2/2)\sin^{-1}(y^{1/3}/d) = (Kx/\sqrt{3}) + C_1$$

Since $y = O$ when $x = O$, $C_1 = O$. Then,

$$x = -\sqrt{\frac{3}{4}\frac{d^2y^{2/3}}{K^2} - \frac{3}{4}\frac{y^{4/3}}{K^2}} + \frac{\sqrt{3}d^2}{2K}\sin^{-1}\frac{y^{1/3}}{d} \tag{6}$$

Another boundary condition, $y = O$ when $x = L$, gives

$$d^2\pi/2 = KL/\sqrt{3};\ d^2/K = 2L/\sqrt{3}\pi$$

From Eq. (4),

$$y^{4/3}/K^2 = EI/P$$

Eq. (6) becomes

$$x = \frac{L}{\pi}\left[-\sqrt{\left(\frac{3\pi^2}{4}\frac{EI}{PL^2}\right)^{1/2} - \frac{3\pi^2}{4}\frac{EI}{PL^2}} + \sin^{-1}\left(\frac{3\pi^2}{4}\frac{EI}{PL^2}\right)^{1/4}\right] \tag{7}$$

The maximum I is required when

$$\sin^{-1}\left(\frac{3\pi^2}{4}\frac{EI}{PL^2}\right)^{1/4} = \frac{\pi}{2}; \qquad \left(\frac{3\pi^2}{4}\frac{EI}{PL^2}\right)^{1/4} = 1$$

So,

$$I_m = (4/3\pi^2)(PL^2/E)$$

In terms of I_m, Eq. (7) is

$$x = \frac{L}{\pi}\left[-\sqrt{\left(\frac{I}{I_m}\right)^{1/2} - \frac{I}{I_m}} + \sin^{-1}\left(\frac{I}{I_m}\right)^{1/4}\right] \tag{8}$$

Eq. (8) is sufficient for the determination of the cross-section at any point. By using Calculus of Variations, Blasius found that this result gives the circular column of minimum volume for a given critical load and length.

It is interesting to compare this with the prismatic column. Consider first the maximum moment of inertia required for the same length and critical load. The coefficient of PL^2/E is given below:

Prismatic column	0.1013
M/EI constant	0.125
Mc/EI constant	0.1352

A saving in material results from using a column of varying section. The ratio of the volume of the varying column to that of the prismatic column for the same length and critical load appears below. Both figures are based on a circular section.

M/EI constant	0.871
Mc/EI constant	0.866

From these figures, it appears that it makes little difference which criterion is used, constant M/EI or constant Mc/EI. The ease of application of the former is certainly a point in its favor.

Centrally Loaded Column of Constant M/EI Under Critical Load P. Ends Restrained

The entire column appears in Fig. 2*a*. The points of contraflexure, or points of zero moment, are marked

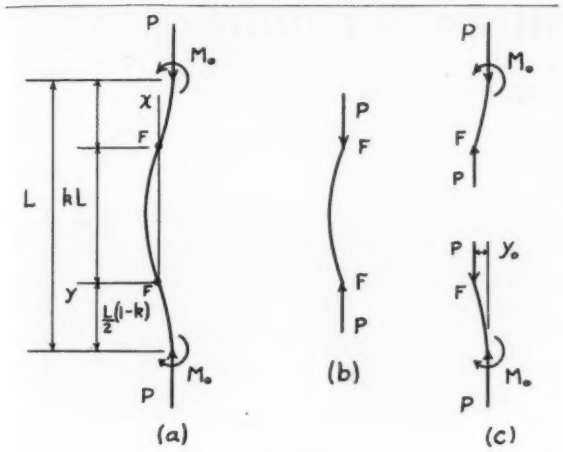

FIG. 2.

F, with the origin placed at the lower one. It is possible to keep M/EI, which equals d^2y/dx^2, of the same numerical value throughout the entire length, with a change of algebraic sign at each point F. For purposes of analysis, it is convenient to consider the column as divided into the three parts shown in Figs. 2*b* and 2*c*. The central part corresponds to the column discussed previously, with a change in length from L to kL. Then Eq. (2) may be applied and the moment of inertia is,

$$I = (P/2E)(kLx - x^2) \tag{9}$$

The maximum moment of inertia is $I_m = Pk^2L^2/8E$. While Eq. (9) gives true values only for the central part of the column, each end part, of length $(L/2)(1 - k)$, may be treated as a reflection about F, of an equal length of the central part. Thus, the required moment of inertia at any point in the column may be obtained in terms of k.

With a stable restraint at the end, $M_0 = n\theta$, where θ is the amount of rotation at the end, and n is a constant of proportionality. The use of n makes it possible to express the end restraint quantitatively. For small deflections, θ may be replaced by the slope, dy/dx. The general expression for the slope is,

$$dy/dx = (A/2)\ (kL - 2x)$$

At the lower end,

$$dy/dx = (AL/2)(2k - 1)$$

Thus

$$M_0 = (nAL/2)(2k - 1)$$

Also, from Fig. 2*c*, M_0 is equal to the product of P and the distance y_0. In general,

$$y = (A/2)(kLx - x^2)$$

At the lower end,

$$y_1 = -\ (A/2)[(L^2/4) - kL^2 + (3k^2L^2/4)]$$

Then,

$$M_0 = -(PAL^2/8)(1 - 4k + 3k^2)$$

Equating the two values for M_0,

$$(nAL/2)(2k - 1) = -(PAL^2/8)(1 - 4k + 3k^2)$$
$$n/PL = (1 - 3k)(1 - k)/(1 - 2k) \tag{10}$$

Eq. (10) is sufficient for the determination of k for a given n.

The moment of inertia equations developed above indicate zero dimensions for the cross-section at certain points. Slight modifications would of course be necessary at these points.

The Measurement of Visibility

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ABSTRACT

Certain fundamental principles involved in the measurement of visibility are discussed in relation to the use of the Luckiesh-Moss Visibility Meter. The construction, calibration and operation of this instrument are described. In view of the variety of visual problems in the aeronautical sciences and practices the present discussion is directed to general rather than specific techniques for investigating individual problems.

VISIBILITY

IF AN OBJECT is visible, it is necessary that the size of the object, its contrast with its background, the brightness of the object or background, and the duration of the image upon the retina each exceed certain individual threshold values.[1, 3] The threshold value of each of these variables is determined by the magnitudes of the other variables of the threshold stimulus. This is equivalent to stating that, within certain limits, a deficiency in one factor may be compensated by an augmentation of one or more of the other factors. It will be noted that visibility, as a measurable quantity, depends not only upon the physical characteristics of the object of regard but also upon the efficacy of the visual processes of the observer. However, if it be specified that all measurements of visibility are to be made by observers possessing normal vision or reduced to such a basis, the measured quantity then denotes a normal attribute of the object of regard expressed upon a rational scale.

A common characteristic of present methods of measuring visibility is that such appraisals are based upon observations made under threshold conditions. This inherent restriction is due to the fact that the only unique visual stimulus between invisibility and maximal visibility is that of threshold visibility. Upon this basis, it is axiomatically assumed (1) that dissimilar objects of threshold visibility are equal in visibility and (2) that the same augmentation of one or more of the fundamental visual variables will cause these objects to be likewise equal in supra-threshold visibility. It is further assumed that the degree of supra-threshold visibility of a given object is proportional to the decrease in the size, brightness, contrast or viewing-time required to reduce the object to threshold visibility.[2] For example, if the factors of brightness, contrast and time remain constant, the degree of supra-threshold visibility of a given object may be measured by the difference between its actual size and that size which would barely afford threshold visibility. This concept of the degree of supra-threshold visibility is analogous to the concept of "factor of safety," as used in engineering. This concept and a means of measuring supra-threshold visibility have been given little attention in a world in which things to be visible with certainty and with ease must be far above the threshold of visibility.[2]

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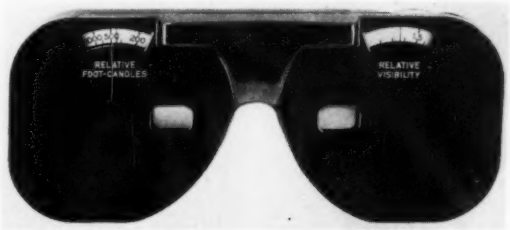

FIG. 1. The Luckiesh-Moss Visibility Meter bearing scales calibrated in "Relative Visibility" and "Relative Footcandles," respectively. The latter scale is not discussed in the present paper. Obviously various other scales of factors which influence visibility can replace the footcandle scale on the left. In fact, data for several such scales are already available.

THE LUCKIESH-MOSS VISIBILITY METER

It follows from these basic principles that a "visibility meter" essentially involves (1) means for reducing the object of regard to threshold visibility and (2) means for expressing this reduction by impairment in the visual stimulus upon an appropriate scale. Thus a visibility meter may be calibrated in terms of a particular fundamental visual variable while the requisite threshold conditions may be attained by altering any one or any combination of these variables. In the case of the meter developed by the authors, threshold conditions are obtained through the simultaneous reduction of the factors of brightness and contrast. The practical advantage of this procedure is the enormous changes in visibility which are readily accomplished. However, the scale of visibility is calibrated in terms of the factor of size. The merit and usefulness of such a scale are largely due to the obviousness of the factor of size in seeing. In fact, size is more readily visualized than the other fundamental factors.

The L-M Visibility Meter, illustrated in Fig. 1, consists essentially of two identical colorless circular gradient filters, precise in optical density and diffusion, which may be rotated synchronously in front of the eyes of the observer while binocularly viewing an object while performing a visual task. One of these filters is illustrated in Fig. 2. These filters incidentally reduce the apparent brightness of the visual field by absorption

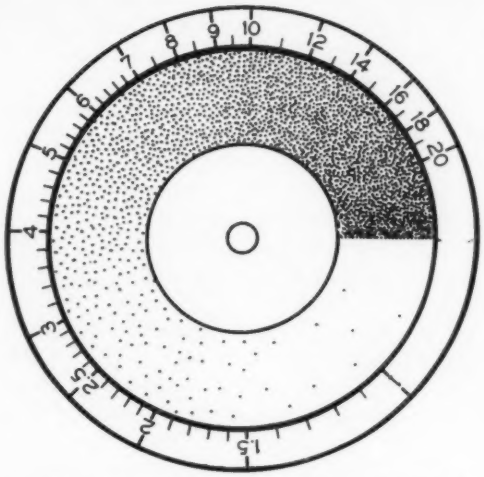

FIG. 2. One of the two gradient filters of the L-M Visibility Meter. The scale shown is calibrated in terms of threshold size as a measure of visibility. The scale-values indicate size in minutes visual angle.

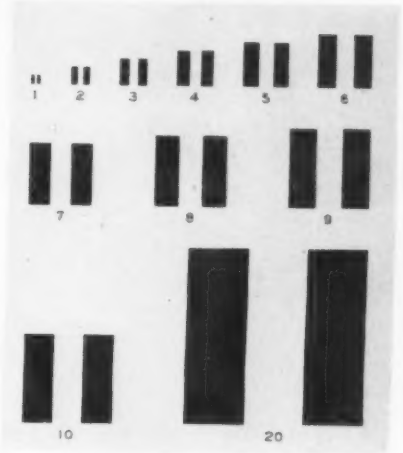

FIG. 3. Reproductions of the parallel-bar test-objects used in calibrating the L-M Visibility Meter. If these objects are viewed from a distance of 5 feet, the numerical values below the test-objects correspond to the visual angles in minutes subtended by the width of the bars or of the width of the space between the bars.

but, of more importance, they reduce the degree of contrast between the object of regard and its background, due to the slightly diffusing character (*cf.* "Q" factor in photography) of the photographic filters. Thus threshold conditions are established by rotating the gradient filters until the object of regard just emerges from, or disappears into invisibility; that is, into the obscurity of the blurred visual field. As yet it has been impracticable to produce filters which vary only the contrast and not the brightness but some promising possibilities are being investigated.

In developing a quantitative scale of visibility, it is recognized that the specification of a unit of visibility is an arbitrary matter. However, this is equally true of such other units as the meter, gram, decibel, lambert, etc. The fact that a unit of visibility necessarily involves many arbitrary specifications does not lessen its practical usefulness. For example, the unit of visual acuity, as used in ophthalmology for rating human vision, involves all of the variables present in the specification of unit visibility as applied to objects. However, many of these variables disappear when visibility is measured upon a relative basis. This is likewise true in ocular refraction when the purpose of the observations is to measure refractive errors rather than to determine the "absolute" visual ability of the patient.

The visibility meter is empirically calibrated by observing a series of test-objects of identical configuration but varying in size from the smallest visible to the largest encompassable by the human fovea.[4] The latter represents a practical upper limit in visibility since the effects of increasing beyond this value the size of a black object on a white background illuminated to a brightness of 10 footlamberts are practically negligible. Such a series of test-objects is illustrated, with some omissions, in Fig. 3. In this case, the "size" of the object is defined as the width of one of the parallel bars or the width of the space between them measured in minutes visual angle. It will be noted that the image of a "20-minute object" occupies a square 60 minutes on each side, or practically the entire area of the fovea. These test-objects were repeatedly viewed by a group of observers possessing normal vision and the mean position of the gradient filter at which each test-object became visible was marked upon the visibility scale (Fig. 2) attached to the filter.

In principle, the threshold visibility of *any* object, viewed under *any* condition by *any* observer is equivalent to one of the infinite series of standard test-objects. A value of "1" on the scale of visibility indicates that a standard test-object, whose critical details subtend a visual angle of one minute, is of threshold visibility when viewed through a region of the filters of the meter so designated. Hence any object which is barely visible through the same portion of the filters is assigned a value of unity in visibility. Specifically, the visibility of such an object, under the conditions of measurement, is equivalent to that of a 1-minute standard test-object as viewed by an observer possessing normal vision and under a brightness-level of 10 footlamberts. The latter object is practically equivalent in visibility to the "20/20" characters (assumed standard for normal vision) on the test-chart of the ophthalmologist. Similarly, a scale value of "2" indicates that the visibility of the object of regard is equal to that of a 2-minute standard test-object or to that of the "20/40" characters of the test-chart, etc. Obviously, these values of visibility are significant in an absolute sense, and they are uniquely related to the universally assumed standard of normal vision, only when the observations

are made under the same conditions as those prevailing during the calibration of the meter.

Since a scale value of "1" represents threshold visibility under the specified conditions of calibration, it is obvious that higher values represent supra-threshold visibilities. For example, if the visibility of a given object is "5," such an object is equivalent in visibility to a standard parallel-bar object five times as large as the smallest visible under the same conditions of observation. Thus the object of regard may be said to possess a visual "factor of safety" of five with respect to the variable of size of an equivalent standard object. However, in the interpretation of such data it should be recognized that the factor of size obeys a "law of diminishing returns" as the object becomes larger. In consideration of this fact another scale of visibility has been developed[2] in which the units of visibility throughout its entire range are equivalent in visual effectiveness.

Visibility Measurements

The visibility meter may be used in practice in two different ways: (1) as a means for measuring visibility in terms of a rational absolute scale; or (2) as a means for appraising the visibilities of two or more objects upon a relative basis. If significance is to be attached to the scale values in an absolute sense it is obvious that all measurements must be made under the same physical conditions as those prevailing during the calibration of the meter. These conditions may be reproduced readily. It is also necessary that the observers shall possess normal vision and the same introspective conception of threshold visibility as the observers used in the calibration of the meter. Since these human variables are usually beyond control, it becomes necessary to obtain "correction factors" for the individual observers. For example, if a 10-minute standard object is found by an individual observer, to possess a visibility rating of "8," all measurements taken by this observer should be increased by 25 per cent. Obviously, such corrections would be required in meteorological surveys conducted by different observers.

The simplest and most general use of the visibility meter is in determining the relative visibilities of various objects. When used for this purpose, the *only* requirement is that the same conditions prevail throughout a given series of measurements. Thus the measurements may be made at any distance, under any level of illumination or lighting condition and by any observer. Furthermore, the values of relative visibility so obtained may be reproduced with considerable accuracy by another observer and under other conditions for observation. For example, a group of children in the sixth grade of a Sight-Saving class obtained practically the same ratio (within less than 2 per cent) in visibility between two typographical samples as did a group of technically trained adult subjects possessing normal or near-normal vision.

It should be recognized that a single measurement whose magnitude may be markedly affected by human vagaries possesses but little intrinsic reliability. Hence only average values, obtained from an adequate number of measurements in each case, should be given quantitative consideration. Obviously, the number of observations required to produce a reliable measure of visibility depends upon the skill of the observer and upon the significance to be attached to the value found. To discriminate between objects which differ but little in visibility, it is necessary to make more observations than are required where the differences are large. In general, it is advisable to test by statistical methods the reliability of all measurements of visibility. For example, if the difference in visibility between two objects is found to be several times as large as its probable error, the difference may be considered to be significant. As experience is gained in the use of the meter, and in interpreting visibility data, there will be less need for detailed statistical analyses of the data. Analyses of the precision obtainable with the L-M Visibility Meter, including a study of the variability within a single series of measurements, as well as the variability among average values obtained on different occasions, are available elsewhere.[3]

Applications in Aeronautical Sciences

The visibility meter should be of value in many ways in aeronautics such as (1) in appraising the influence of fog, smoke, etc., upon the visibility of appropriately placed standardized targets, (2) in measuring the loss in visibility due to transparent media, windows, goggles, etc., in the line of vision, (3) in determining the visibility of various signals, different types of instrument dials, etc., and (4) in appraising the visual abilities of pilots and observers to distinguish low contrasts at night as well as during the day. Some of these purposes require absolute measurements involving the observation of standardized test-objects under rigidly specified conditions, while others require merely relative measurements under any constant set of conditions. An appreciation of the principles involved in the measurement of visibility should provide an adequate basis revealing uses for the meter and for developing techniques of measurement in specific fields of application. Obviously knowledge in the fields of seeing and of aids to seeing can only be established by meaningful measurements as in other fields.

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The Transmission of Light in the Atmosphere with Applications to Aviation

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ABSTRACT

The transmission of light through the atmosphere is determined by the absorption and scattering which it suffers. It is shown that absorption is negligible in the visible spectrum but becomes increasingly strong in the infrared. Scattering is due to suspended particles ranging in size from the air molecules to rain drops. The scattering coefficient changes with the wave length, or color, of the light only when the scattering particles are smaller than the wave length. Since the suspensoids which produce low visibility are large compared to the wave length of visible light and the near infrared, no advantage results from the use of colored lights or the infrared in such cases. Photography in the near infrared is useful in cases when the visibility is limited by small particle haze.

Theory shows that the visual range is determined largely by the scattering coefficient of the atmosphere. Since it is often difficult to select suitable marks for the estimation of the visual range, it is suggested that an instrument be developed for determining the scattering coefficient.

INTRODUCTION

ALTHOUGH THE transmission of light through the atmosphere has long been of interest to astronomers and other workers in the natural sciences, the advent of air transportation has served to emphasize the practical importance of this subject. In spite of the great advances in navigation instruments, pilots still depend to a considerable extent on visual observation of prominent landmarks by day and of lights and beacons at night. The distance at which such marks can be seen is largely determined by the transparency of the atmosphere. The same factor is also of major importance for aerial photography and for military reconnaissance and bombing missions. In an attempt to extend the useful range of observation, lights of various colors and instruments utilizing infrared radiation have been tried with varying degrees of success.

It is the purpose of this paper to discuss the principles governing the transmission of light in the atmosphere and to apply them to some of the problems of special interest for aviation. In particular the factors governing the visibility of objects by day and lights at night will be discussed and also the merits of infrared or other invisible radiations for signalling and photography. No attempt will be made to present a complete discussion of the theory of the visual range nor of the details of photography and special instruments, since this is available elsewhere.

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THE DIMINUTION OF LIGHT IN THE ATMOSPHERE

The most useful measure of light for the purposes of this paper is the flux density which is flux per unit area. Flux is a measure of the visible power, but the concept of flux density may also be used in the case of invisible radiation, in which case the flux is considered to be the total radiant power.

If a beam of light of restricted angular extent is directed through the atmosphere, the flux density of the beam will decrease with distance from the source due to three effects. The first of these is the inverse square law which states that the flux density will decrease inversely as the square of the distance from the source. This decrease is due solely to the geometry of the optical system. Second, the flux density will decrease due to the absorption of the light by the atmospheric gases and suspended matter. Finally, there will be a decrease due to scattering by the air molecules and the various atmospheric suspensoids. In this paper the term scattering is taken to mean any process which changes the direction or spatial distribution of the light and hence includes reflection, refraction and diffraction. Although scattering does not involve any energy transformations, it is the principal factor governing the transmission of visible light in the atmosphere.

ABSORPTION IN THE ATMOSPHERE

The absorption due to the fixed gases of the atmosphere is entirely negligible in the visible and infrared regions of interest to this discussion. Oxygen in both the atomic and molecular forms and ozone absorb strongly in the ultraviolet and combine to set a lower limit of the solar spectrum at about 0.3μ, but this is not of much importance in the present investigation. The important absorbing gases in the atmosphere are carbon dioxide and water vapor. Carbon dioxide has three principal absorption bands, one at about 2.7μ, a second at 4.5μ and the third and most intense extending from about 12 to 16μ. Carbon dioxide absorption thus becomes of significance only in the infrared. Water vapor is also quite transparent within the visible spectrum (.4 to $.7\mu$) but absorbs in several regions in the near infrared and has a complex and intense absorption spectrum in the region beyond 5μ. The general features of this portion of the spectrum are intense absorption from about 5 to 6μ, a relatively transparent region from about 9 to 15μ and rapidly increasing absorption for wave lengths greater than 20μ.

Certain suspensoids, such as smoke and dust, may absorb visible light to some extent, but the scattering caused by such particles appears to be large compared to the absorption. Water drops which occur in the atmosphere as fog, clouds, drizzle, rain, etc., are essentially transparent in the visible spectrum. However, the absorption of water increases rapidly in the infrared, a layer 1 mm. thick being opaque to wave lengths greater than 2.4μ. Because of the effects of scattering, it cannot be assumed that a cloud absorbs the same fraction of the radiation as an equivalent sheet of water, but experiments indicate that an average cloud 100 meters or more in thickness acts essentially as a black absorber in the infrared.

Scattering in the Atmosphere

It is clear from the above discussion that absorption in the atmosphere is important only in the infrared region. The major factor which determines the transmission of visible light in the atmosphere is scattering by the various suspended particles. In approximate order of increasing size of the particles the important scattering elements in the atmosphere are the air molecules themselves, smoke, haze, dust, cloud and fog particles and the various precipitation elements.

In the case of a parallel beam, the reduction in flux density due to scattering as a function of distance is given by the well-known exponential law:

$$E/E_0 = e^{-kx} \tag{1}$$

where E is the flux density at the distance x along the beam
E_0 is the flux density for $x = 0$
k is the scattering coefficient

Since parallel beams are not obtainable in practice, the inverse-square law must be combined with the above exponential expression, giving

$$E/E_0 = I_0 e^{-kx}/x^2 \tag{2}$$

where I_0 is the candlepower of the source, or the beam candlepower in the case of a collimated source.

In general, the scattering coefficient is a complex function of the ratio of the particle diameter and the wave length of the light. For the special case of particles small compared to the wave length Rayleigh[1] showed that the scattering coefficient varied inversely as the fourth power of the wave length. This relation has often been used erroneously to predict the scattering due to particles which are not small compared to the wave length. Table 1 gives the approximate range of "diameters" of the several types of scattering particles in microns. The visible spectrum extends from about .4 to .7μ and the near infrared from 17 to 20μ, according to Strong's[2] classification. It is clear from the table that for visible light, Rayleigh's inverse fourth power relation is valid only for scattering due to molecules and the smaller haze, dust and smoke particles. It is also evident that it would be necessary to go far into the infrared to obtain a wave length large compared to fog and cloud particles. In this region the absorption of water vapor becomes very intense so that the reduction in scattering would probably not result in an increased transmission.

Table 1

Type of Particle	Approximate Diameter of Particles
molecules	10^{-4} microns
smoke	.01–.5
haze	.1–10
dust	.1–50
cloud	5–50
fog	5–100
drizzle	10^2–5×10^2
rain	5×10^2–7×10^3

The variation of the scattering coefficient when the particles are not small compared to the wave length has been treated by Stratton and Houghton[3] for the case of spherical water drops in air. The scattering coefficient was found to be $\pi n r^2 K$ where r is the radius of the drops, n the number of drops per unit volume of air and K is a function of $2\pi r/\lambda$ as shown in Fig. 1. In the original paper the scattering coefficient was given as $2\pi n r^2 K$, but it now appears, as the result of some experimental work, that the factor 2 should not be included. This disagreement is not due to a numerical error but to something more fundamental which is not yet completely understood.

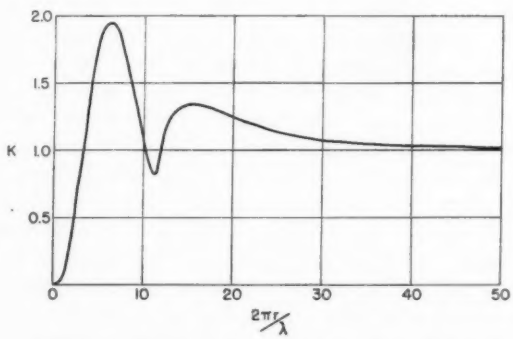

Fig. 1. Curve of K, the ratio of the scattering coefficient to the projected area of the drops for spherical water drops in air, *versus* $2\pi r/\lambda$.

The portion of the curve of Fig. 1 for $2\pi r/\lambda$ less than about unity corresponds to the region of Rayleigh scattering. For large values of $2\pi r/\lambda$, exceeding about 30, K is approximately unity. It is clear that a rapid decrease in scattering with increasing wave lengths can be expected only along the steep portion of the curve to the left of the first maximum, that is for wave lengths exceeding the radius of the particles. With the aid of Table 1 the minimum wave length for each type

of suspensoid may be determined. It is clear that no great reduction in the scattering produced by particles as large as fog and cloud drops can be expected in the visible spectrum or in the near infrared. This result has been confirmed by recent measurements of Sanderson.[4]

Although Fig. 1 applies strictly only to spherical, non-absorbing water drops, it is believed that it is also approximately correct, at least in the regions to the left of the first maximum and beyond the second maximum, for non-spherical particles of approximately the same index of refraction. It will be noted that as K approaches unity the scattering coefficient πnr^2K approaches the total projected area of the drops in a unit volume of air. That is, all of the light intercepted by the drops is scattered. It would appear from geometrical optics that this would hold for non-absorbing particles of any shape and index of refraction as long as the particles are large compared to the wave length, although the spatial distribution of the scattered light will be different in each case. If the particles also absorb, it is not proper to add the scattering and absorption coefficients. If the absorbing particles are large compared to the wave length, it would appear that the total of the absorption plus scattering could not exceed the scattering due to a non-absorbing particle of the same size. It is not possible to predict the fractions scattered and absorbed except in special cases. The case of absorbing particles which are not large compared to the wave length is very complex and no complete solution to this problem has been found.

Applications to Light Signals

In the case of visual signal lights or beacons, it is clear from the above general discussion that no change in transmission with wave length can be expected unless the scattering particles are small compared to the wave length. Low visibility is generally due to particles larger than the wave length of visible light, and experience as well as theory has demonstrated that colored lights are of no advantage in such cases. Occasionally a fine-particle haze may reduce the visibility to from three to five miles. If it is assumed that Rayleigh's inverse fourth power law applies, a relatively small change in wave length will produce a considerable change in transmission. However, the sensitivity of the eye, which is a maximum at about .55 also varies rapidly with wave length. These two effects will combine to give a maximum visibility in the neighborhood of 0.6μ, which corresponds approximately to light from a sodium lamp. However, this conclusion applies only to the special case of a haze composed of very small particles and, in general, no appreciable gain can be expected as has been shown experimentally by Luckiesh and Holladay.[5] In particular, color filters over white lights definitely reduce the visibility of the light, since a filter decreases the intensity of the source even in the pass band of the filter. Colored or infrared lights may sometimes be necessary for purposes of identification or secrecy, but no appreciable gain in visibility is produced even when the lights are viewed against a general background illumination as shown by Langmuir and Westendorp.[6] These authors also found that the visibility of flashing point sources never exceeds that of a steady light of the same intensity.

As shown in Eq. (2), the flux density at a distance x from a point source of intensity I is directly proportional to the intensity and to the transmission of the atmosphere and inversely proportional to the square of the distance. The light will be visible if the flux density at the eye exceeds a certain minimum value which may be taken as about 2×10^{-11} lumens per sq. cm.[7] The candlepower required to make the light visible at a distance x is then

$$I = E_0 x^2 e^{kx} \tag{3}$$

where E_0 is the threshold flux density for the eye. As will be indicated below, the daytime visual range of a dark object against the horizon sky is given by

$$x_m = (\log_e 1/\epsilon)/k \tag{4}$$

where ϵ is the threshold of contrast which has a value of about .02. Thus, with the aid of Eq. (4), Eq. (3) may be written

$$I = E_0 x^2/(\epsilon)^{x/x_m} \tag{5}$$

As an example of the application of this formula suppose that the transmission of the atmosphere is such that the daytime visual range would be one mile. It is desired to find the candlepower of a light required to make it visible at two miles.

$$I = [2 \times 10^{-11} \times (3.22)^2 \times 10^{10}]/(.02)^2$$
$$= 5190 \text{ c.p.}$$

The only practicable means for increasing the visual range of a light is to increase its candlepower in accordance with Eq. (5). As shown by the above example, it is relatively simple to provide lights which can be seen at night at distances considerable greater than the daytime visual range in the same atmosphere.

Applications to Photography

The availability of films which are sensitive in the near infrared has made it possible to take advantage of the fact that on days with relatively good visibility the scattering particles are usually small compared to the wave length of visible light. In such cases, the atmosphere may be several times as transparent in the near infrared as in the visible. It also follows that the veiling glare due to the scattered light will be correspondingly reduced in the infrared. These factors combine to yield much clearer photographs in the infrared. Since infrared photography is limited to the very near infrared (limit about 1.2μ), it will not permit pho-

tographs to be taken through fog or other suspensoids in which the particles are not less than about 0.5μ in diameter. Unless distant objects appear bluish, it is extremely doubtful if infrared photography will be advantageous. For a detailed presentation of the many problems involved in infrared photography, the reader is referred to the recent book by Greenwood.[8]

Daytime Visibility

In the absence of color contrast, an object is visible when the ratio of the difference in brightness between the object and its background to the background brightness exceeds a certain minimum value. The value of this brightness threshold is .01 to .02 under normal conditions, although it is subject to considerable variation. As the distance between an observer and the object increases, the apparent brightness of a dark object increases due to the sun and sky light which is scattered into the eyes of the observer by the atmospheric suspensoids in the line of sight. A distance will finally be reached at which the apparent brightness of the object will approach that of the background so closely that the eye cannot distinguish the object. This limiting distance which is called the "visual range" or the "visibility" is a function of the scattering coefficient of the atmosphere and of the reflectivity of the object and the background. Expressions for the visual range for a number of cases have been derived but only the results will be presented here. For more details the reader is referred to the excellent book by Middleton.[9]

Case 1. A black object viewed against the horizon sky.

$$x_m = (\log_e 1/\epsilon)/k$$

where x_m = the visual range
ϵ = contrast threshold (.01 to .02)
k = scattering coefficient

Case 2. An object of albedo* A viewed against the horizon for a totally overcast sky.

$$x_m = (1/k) \log_e (1 - A/2)/\epsilon$$

Case 3. Object of albedo A_1 viewed against a background of albedo A_2. Totally overcast sky. Object and background at the same distance.

$$x_m = (1/k) \log_e (1/2\epsilon) \left[(A_1 - A_2) + 2\epsilon - \epsilon A_2\right]$$

If $(A_1 - A_2)$ is reasonably large, the latter two terms in the brackets may be neglected.

An inspection of the above formulas shows that in each case the visual range is inversely proportional to the scattering coefficient. The expression given above for Case 1 has been experimentally verified by Hulburt.[10]

The ratio of the visual range of an object of albedo A to that of a black object when both are seen against the horizon sky under the same conditions is

$$\log \left\{[1 - (A/2)]/\epsilon\right\}/\log (1/\epsilon)$$

The minimum value of this ratio is .82, which corresponds to the maximum possible albedo of unity. It follows that as long as objects are viewed against the horizon sky, the albedo of the object has little effect on the observed visual range, although dark objects should be selected if possible. When the object is viewed against a background of albedo A_2 (Case 3), the visual range is almost directly proportional to the log of the difference in albedos of the object and the background. It is clearly possible for the visual range of an object seen against a terrestrial background to be considerably less than the visual range of the same object seen against the horizon sky. The maximum possible value of $(A_1 - A_2)$ is unity, and for this case the visual range is again 82 per cent of the visual range of a black object against the horizon sky. This is of considerable importance in aviation, since visibility estimates are usually based on observations of objects seen against the sky while a pilot views objects against the ground.

If the scattering coefficient k could be assumed constant in all directions, it would be possible to obtain a relation between the visual range of a black object seen against the sky and an object of known albedo viewed against the ground. The suspensoids which determine the scattering coefficient in the lower atmosphere are usually stratified horizontally and decrease rapidly in number with altitude. This is borne out by the exceptional horizontal visual ranges often observed at high levels. Thus the mean scattering coefficient in the vertical direction is commonly considerably less than in the horizontal direction at the ground. This difference is often so marked that a well-defined "cone of visibility," with its apex at the object, is formed when an object on the ground is viewed from the air. The included angle of the cone will usually decrease as the elevation of the point of observation is increased.

Airplanes are often visible from the ground when the pilot is unable to observe any objects on the ground. As often suggested, this may be due to reflection of sunlight from the sharp "top" of a haze layer which may form at a temperature inversion. However, even in the absence of this phenomenon, the visual range of an airplane will often exceed that of terrestrial objects because of the difference in background (Case 2 *vs.* Case 3). This would always be true except that the contrast threshold increases as the angle subtended by the object at the eye decreases and prominent terrestrial objects are usually larger than airplanes.

In connection with camouflage problems, it should be noted that an object can never appear as bright as the sky unless it is reflecting sunlight. Furthermore, a "black" airplane would be visible about 20 per cent further away than one having a diffusely reflecting surface of albedo unity.

* Albedo = diffuse reflection coefficient.

Measurement of the Visual Range

It is customary to estimate the visual range by noting the most distant prominent object which can be seen. It is seldom possible to select a series of marks located at appropriate distances which are all seen against the horizon sky, are all nearly black and of sufficient size. These difficulties could be overcome by the use of a suitable instrumental technique. This would have the further advantage of making measurements at different stations directly comparable.

The most useful quantity to measure is the absorption coefficient, since it would then be possible to interpret this in terms of the visual range either of lights of known candlepower or of objects seen either against the sky or the ground. It would also be desirable to know the variation of the absorption coefficient with elevation so that the oblique visual range could be estimated, but an accurate determination at the ground level should be attempted first.

The scattering coefficient may be determined most directly by measuring the transmission of light over a path of known length,[11] but the equipment is too expensive for general use. A number of empirical meters have been developed in Europe in which ground glass discs of various degrees of roughness are interposed between the eye and the object until the object is no longer visible.[12] These instruments do not measure the scattering coefficient and they depend to some extent on the human equation. In spite of these limitations, they have given good results but have not been used in this country. It is suggested that a more suitable instrument would be a visual photometer adapted to measuring the difference in brightness of a relatively dark object at a known distance and the horizon sky. Such a measurement would determine the scattering coefficient directly. Goldberg[13] has invented a photometer which could be used for this purpose, and it is believed that a simpler and inexpensive photometer of this type might be developed. Such an instrument could also be used at night if arranged so that a distant light of known candlepower could be compared with a standard light in the instrument as was done by Middleton.[7]

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An Advanced Method for the Determination of Intermediate Wing Contours

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The Glenn L. Martin Company

SUMMARY

As speeds are approaching critical values and the use of twisted and dis-similar airfoil sections is becoming more frequent, the accuracy of the full scale master lines from which wings are constructed is becoming more critical. A geometrical method is suggested for the determination of intermediate wing contours for wings with dis-similar or twisted airfoils.

The problem, briefly stated, is to determine intermediate wing contours when the root and tip airfoil sections and their relation to each other are given. It is assumed that the most desirable wing aerodynamically and structurally, where there is no reverse of curvature in the limiting airfoils, is a straight line generated surface about these two limiting sections. (Where there is reverse of curvature in just one of the airfoils, a compromise is desirable.)

LIMITATIONS AND ERRORS OF CUSTOMARY METHOD

THE CUSTOMARY method of determining intermediate contours involves the selection of points on the limiting airfoils at corresponding percentages of the airfoil chords and the drawing of straight lines between these points, which straight lines are then divided proportionally to the wing rib stations to give points for the drawing of the intermediate contours (see Fig. 1).

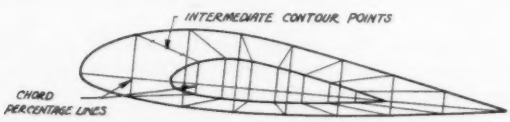

FIG. 1.

This method arose from the much used straight line tapered wing having similar airfoils with chords in a common plane. For this special case the percentage lines are straight lines on the generated surface. However, as soon as the limiting airfoil sections are made dissimilar or are twisted with reference to each other, then the percentage lines are no longer the location of the generating lines and the percentage points on the generated surface form curved lines. So far, the error involved in assuming the percentage lines as straight lines and forcing the wing skin to fit the contours thus determined has not proved of practical importance. Nevertheless, wing ribs made from contours determined by this method will force a series of flats or reverses in fairing in the wing skin and as the airfoil dis-similarity and wing twist increase, this forcing will reach a structural limit, especially with corrugated wing box construction. Furthermore, it is believed that even very slight surface irregularity will be aerodynamically prohibitive in the future.

TANGENTIAL METHOD OF CONTOUR FAIRING

A uniform sheet wrapped around any two limiting smooth contours will have straight lines on its surface in certain positions only. Fig. 2 illustrates this fact; the dotted lines on this figure connect the conventional percentage points, the full lines show the location of straight lines as determined by the use of a straight edge. In this case it is apparent that the assumption of straight lines along the percentage points is in error.

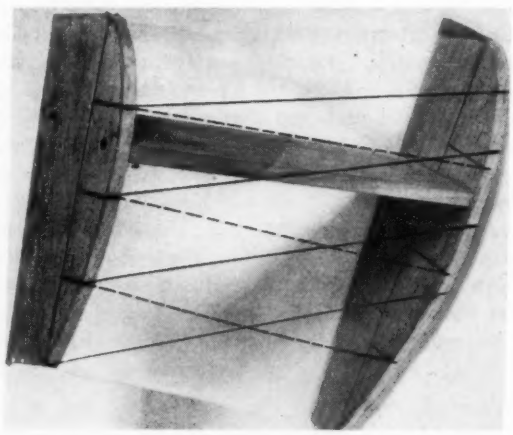

FIG. 2.

To determine the location of the straight lines (generatrices) on our surface let us place the limiting airfoils in parallel planes A and B and lay our surface on a third plane D (see Fig. 3). It becomes apparent that the surface makes line contact with plane D through points a and b on the two limiting airfoils where the airfoils are tangent to the plane. Since these tangents lie in a common plane in one projection and in parallel planes in another projection, then in any other projection they will appear as parallel lines. Thus, the criterion for the location of the generatrices is that they connect points of parallel tangency on the limiting

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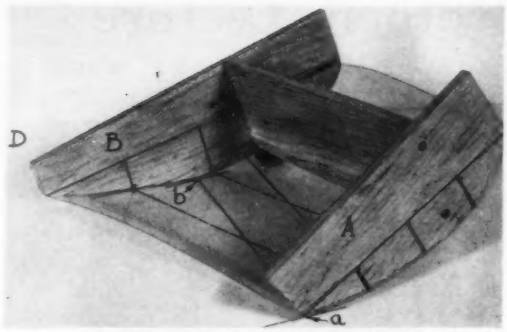

FIG. 3.

curves. For a more rigorous proof of this conclusion see ref. 1.

As an example, Fig. 4 is an exaggerated wing with dis-similar and twisted limiting airfoils *A* and *B*. By constructing parallel tangents such as *ab* and *cd* the generatrices are readily located around airfoil *A* to point *l*. This leaves area *lnm* for which we have points on airfoil *B* but no corresponding points on airfoil *A*.

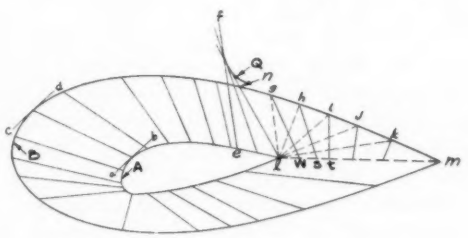

FIG. 4.

In the practical case of an actual skin wrapped around airfoils *A* and *B* the surface *lmn* would contain straight lines in some such locations as *gw*, *hs*, *it*, etc., and would terminate in a curved trailing edge. However, for structural reasons a straight trailing edge is desirable and so it is best to consider the area *lnm* as a portion of a cone with apex at *l* and draw generatrices of the cone as shown by the dotted lines in Fig. 4.

The envelope of the generatrices should fall on a smooth curve called the cuspidal (*Q*, Fig. 4). The cuspidal curve is useful as a check on the proper location of the generatrices and there should be such a curve asymptotically approaching each generatrix where the rate of curvature of the two limiting curves becomes equal. The location of this asymptote may be of aerodynamic use in determining the location of points of equal pressure on the wing surface.

CONCLUSIONS

1. The percentage line method for the determination of intermediate wing contours is accurate only for straight taper wings with similar limiting airfoils having chords in a common plane.

2. For the determination of correct intermediate contours on wings having dis-similar limiting airfoils or wing twist, the following procedure is suggested:

(*a*) Lay out the two limiting airfoil sections to full scale and in their correct angular relation.

(*b*) Through arbitrary points on the limiting airfoils draw generatrices by the tangential method.

(*c*) Divide the generatrices proportionally to the spanwise locations of the intermediate contours desired and draw intermediate contours through these points.

3. For the application of the tangential method to control surfaces, see ref. 2.

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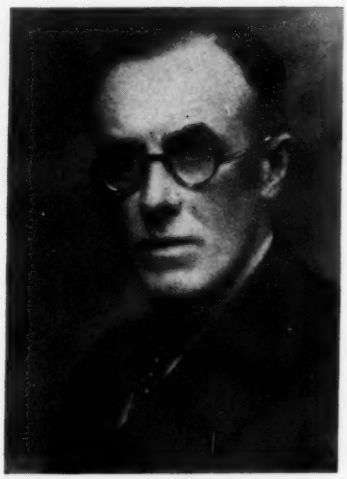
The Fifth Wright Brothers Lecture

THE WRIGHT BROTHERS Lecture, commemorating the thirty-eighth anniversary of the first flights made by Orville and Wilbur Wright at Kitty Hawk, was presented by the Institute on Wednesday, December 17, 1941, at Columbia University, New York City. This was the fifth of these annual lectures which are endowed by the Vernon Lynch Fund established in 1937 by the late Edmund C. Lynch in memory of his brother.

The lecturer was Richard V. Southwell, Professor of Engineering Science at Oxford University, England. Professor Southwell holds an M.A. degree from Oxford University and also from Cambridge University where he was one time lecturer in Mathematics. He is a world renowned specialist in aerodynamics, hydrodynamics, theory of structures, elasticity and mathematical applications to aeronautical problems.



Prof. Richard V. Southwell

DINNER IN HONOR OF THE LECTURER

Presentation of the lecture was preceded by a dinner in honor of Professor Southwell, held at 6:30 p.m. at the Men's Faculty Club of Columbia University. It was attended by 200 Institute members and their guests.

Frank W. Caldwell, President of the Institute, acted as Toastmaster.

Mr. Caldwell introduced several of the distinguished guests. In presenting the guest of honor, he said:

"Professor Southwell, I wish to express our appreciation of your making this trip to America to address the Institute on the occasion of the Wright Brothers Lecture. I also wish to welcome you to America. We had expected to welcome you as a distinguished British scientist, as a representative of British aeronautical science and of the Royal Aeronautical Society. Tonight, we can go further. As American citizens, we can welcome you as a representative of a staunch ally."

PRESENTATION OF THE LECTURE

The lecture was held at 8:30 p.m. before a capacity audience in the main lecture theatre of the Pupin Physics Laboratories Building of Columbia University. Chairman of the meeting was Dr. J. C. Hunsaker, Chairman of the National Advisory Committee for Aeronautics and Past President of the Institute. Honorary Chairmen were C. R. Fairey, Deputy Director-General of the British Air Commission in Washington and Past President of the Royal Aeronautical Society of Great Britain, and Dr. Charles Darwin, Chairman of the British Central Scientific Office in Washington and Director of the National Physical Laboratory, England. Mr. Caldwell opened the meeting and introduced Dr. Hunsaker.

Dr. Hunsaker: "We meet tonight on the 38th anniversary of the Wright Brothers' first flight over the beach at Kitty Hawk, North Carolina, in 1903, and for the fifth Wright Brothers Lecture. To mark this event, and the most important invention of the century, the Royal Aeronautical Society of Great Britain has held for many years at London the Wilbur Wright Memorial Lecture. Some twenty years ago, I was honored by an invitation to give this lecture, and was most hospitably and kindly helped through what seemed to me a pretty difficult assignment.

"I had hoped to welcome to this meeting as an Honorary Chairman, Mr. C. R. Fairey, Past President of the Royal Aeronautical Society and a member of the Council of the oldest and most distinguished aeronautical society in the world. Mr. Fairey is in our country in connection with the joint effort to create fleets of airplanes adequate to our grave needs and was at the last moment prevented from coming. However, I have a cablegram from Mr. Griffith Brewer, who is so ably leading the activities of the Royal Aeronautical Society in London today:

"On behalf of my council and all members, I convey to you on the occasion of the reading of the Wright Brothers Lecture by our distinguished member, Professor Southwell, our best wishes for a successful meeting. On this particular occasion I would add how profoundly we are all moved by the fact that members of the two societies are now in it all the way until the freedom of mankind and peaceful flight are assured to us all.

GRIFFITH BREWER, *President*
The Royal Aeronautical Society

"I now wish to call upon Lester D. Gardner, Executive Vice-President of the Institute, who has some announcements to make."

Mr. Gardner: "At this first meeting of Institute members since the passing on December 9 of our Benefactor Member, Hart O. Berg, I want to take the opportunity to express publicly the great personal loss that is felt by the Officers and Council of the Institute and many members who knew him personally. He will be missed at our meetings where his gracious courtesy was always in evidence. Hart O. Berg was the one who really started the idea of the Institute having an Aeronautical Archives through his gift of a priceless collection of papers, pamphlets, books of clippings and other original material on the early work of the Wright brothers. His donation made our Archives the most complete and unique depositary of Wright aeronautica in existence.

"When the Institute first considered the inauguration of an annual lecture here, comparable to the Wilbur Wright Memorial Lecture presented each year by the Royal Aeronautical Society in England, we naturally conferred with Mr. Orville Wright to determine his wishes and opinion on the subject. He expressed his approval of our plan to establish the Wright Brothers Lecture and attended the first of these lectures which was given here on December 17, 1937, by Professor B. Melvill Jones, another of our distinguished English members. As you all know, Mr. Wright very seldom makes public appearances or statements of any kind, particularly statements as to his opinions on any public matters—aeronautical or general. Therefore, I feel that the Institute is signally honored to have received from Mr. Wright the following telegram sent to us on this occasion commemorating the thirty-eighth anniversary of the first flights at Kitty Hawk:

> *"Please extend to Professor Southwell my cordial greetings and an expression of my belief that the aeroplane which, in the hands of barbarous men, has been doing England and the world inestimable harm will yet be instrumental in establishing for the entire world a guarantee of human liberty and of lasting peace.*
>
> ORVILLE WRIGHT

"I now have another announcement which I feel is appropriate to make before this gathering of members and friends of the Institute who are all greatly interested in every forward step made by our organization. Two years ago, Mr. Paul Kollsman, a Fellow of the Institute, and the Square D Company of Detroit, gave us $50,000 to be spent on the establishment and maintenance of an aeronautical lending library. The Paul Kollsman Library has rendered such effective aid in lending books by mail to aeronautical students and technical specialists all over the country that Mr. Kollsman has decided to establish this service on a permanent basis. He informed me this week that he will make an additional gift to the Institute so as to establish a permanent fund of $65,000 to be used for the maintenance of the library. I am sure that all of you here join with me in expressing to Mr. Kollsman the appreciation of the Institute for his great generosity in endowing an activity which is so valuable to American aeronautics in these critical times."

Dr. Hunsaker: "We have the honor to have present the official representative of British science and of her learned societies, Dr. Charles Darwin, Chief of the British Central Scientific Office in Washington. Dr. Darwin implements the close collaboration and exchange of ideas between research workers of this country and those of the British Empire, which is proving so effective in the creation of new weapons and techniques."

Dr. Darwin: "I hardly need to tell you my pleasure at being able to be with you tonight and the distinction that I feel has been conferred upon me by the appointment as Honorary Chairman of this Wright Brothers Lecture. When I came to this country nine months ago in connection with my work in Washington, I came as a friend. When my duties take me back to England, as they will in the near future, I am glad that I shall leave you not only as a friend, but as an ally."

Dr. Hunsaker: "Now a word of introduction for our guest, Professor Southwell. You already know he is important to us because he was invited to cross the Atlantic to speak to us. You can also realize this winter journey was not undertaken in response to a mere social invitation. The authorities, including Dr. Darwin, who control such journeys made the visit possible.

"Professor Southwell brings to any problem a broad experience in practically all of the aeronautical sciences. He illustrates the ideal definition of a Fellow of this Institute. He has been for many years a member of the British Aeronautical Research Committee and more recently has been directing the work of the Civil Defense Research Committee. His many contributions to the development of engineering science have been characterized by the approach of the analyst with fluent command of the known mathematical tools, and at the same time with a clear recognition of physical realities.

"The Institute welcomes Professor Southwell as its 1941 Wright Brothers Lecturer, and I am honored to present him to you now."

Professor Southwell expressed his thanks to the Chairman and to the Institute for the opportunity to come to the United States to give his paper and referred to previous pleasant visits here. He then presented the Fifth Wright Brothers Lecture.

Institute Notes

Officers Elected for 1942

Hall L. Hibbard to Be President

The Council has elected Hall L. Hibbard, Vice President and Chief Engineer of Lockheed Aircraft Corporation, as President of the Institute for 1942. He will be inducted into office at the Honors Night Dinner held in New York on January 27.

Mr. Hibbard is responsible for the design of a long series of the noted Lockheed airplanes. He helped in pioneering the development of the twin tail and the single-spar all-metal airplane wing. His designs have used increasingly heavy wing loadings and include an experimental tail-first or canard type of airplane. In the development of air transport types and the Army's famous stratosphere airplane, his work has been outstanding.

Hall L. Hibbard was born in Fredonia, Kansas, and spent his boyhood in the Philippine Islands, where his father was a missionary for 41 years. After graduation from the College of Emporia, he took a postgraduate course in aeronautics at the Massachusetts Institute of Technology and received an M.A. degree in aeronautical engineering in 1927. He became Chief Engineer of the Lockheed company in 1932, having previously been on the engineering staff of Stearman Aircraft Company since his graduation.

Other officers elected by the Council are as follows:

VICE-PRESIDENTS

C. S. Jones, President, Casey Jones School of Aeronautics

Grover Loening, Consulting Aeronautical Engineer

J. G. Vincent, Vice-President of Engineering, Packard Motor Car Company

Thomas A. Morgan, President and Director, The Sperry Corporation

EXECUTIVE VICE-PRESIDENT—Lester D. Gardner

SECRETARY—C. E. Sinclair

TREASURER—Earl D. Osborn, President, Edo Aircraft Corporation

ASSISTANT TREASURER—George R. Forman

CONTROLLER—Joseph J. Maitan

Fellows for 1941 Elected

The following members were elected Fellows of the Institute in 1941:

George W. Brady, Chief Engineer, Curtiss Propeller Div., Curtiss-Wright Corp.

John W. Crowley, Jr., Sr. Aero. Engineer, Langley Mem. Aero. Lab., N.A.C.A.

Ralph S. Damon, Pres., Republic Aviation Corp.

William K. Ebel, Vice-Pres. in charge of Engineering and Chief Engineer, The Glenn L. Martin Co.

Michael E. Gluhareff, Chief of Design and Aerodynamics, Vought-Sikorsky Aircraft Div., United Aircraft Corp.

R. M. Hazen, Vice-Pres. and Chief Engineer, Allison Div., General Motors Corp.

Comdr. F. W. Pennoyer, Jr., U.S.N., Head of Engineering Div., Bureau of Aeronautics, Navy Department.

Harry A. Sutton, Aero. Engineer, Consolidated Aircraft Corp.

Capt. W. W. Webster, U.S.N., Manager, Naval Aircraft Factory, Philadelphia.

Robert J. Woods, Chief Engineer, Bell Aircraft Corp.

Qualifications required of a Fellow of the Institute are that he shall have attained a position of distinction in aeronautics and have made notable and valuable contributions in one of the aeronautical sciences or aeronautical engineering. Not more than ten members of the Institute may be elevated to this grade of membership each year.

Selection of those to receive this honor is made by the present American Fellows and Honorary Fellows of the Institute. Nominations are submitted by them and Fellows selected from this list by secret elimination ballots.

New Corporate Members

The Institute is pleased to announce the affiliation of two nationally known financial institutions which became Corporate Members at the beginning of 1942:

The Chase National Bank of the City of New York, which has for some time been cooperating with all branches of the aviation industry and airline operators. It has made loans for the expansion of plant facilities, working capital needs, and for the acquisition of transport equipment.

The National City Bank of New York, which has long been actively identified with the aircraft industry.

The Institute appreciates the support and interest of these organizations in its work and will be able to assist them, as it does all its Corporate Members, to keep in closer touch with the entire aviation industry and its rapidly changing technical aspects.

Kollsman Fund Made Permanent

At the Wright Brothers Lecture on December 17 it was announced that Paul Kollsman, Fellow and Benefactor Member of the Institute, had made an additional gift to the Institute to set up a permanent endowment fund for the support of The Paul Kollsman Library.

Paul Kollsman

In 1940 Mr. Kollsman and the Square D Company of Detroit gave $50,000 to the Institute to establish a library to lend books to members and others wishing to increase their knowledge in any of the aeronautical sciences. Now Mr. Kollsman has given the Institute an additional $25,000, which, with the remainder of the original fund, will provide a permanent endowment fund of $65,000 to support the activities of The Paul Kollsman Library.

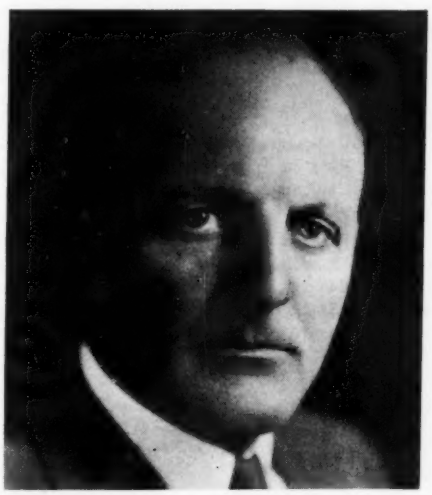
The widespread use of the library since its establishment,

through the hundreds of aeronautical books that have been loaned by mail without charge to aeronautical students and engineers throughout the country, attests to the value of the contribution that Mr. Kollsman's generous gifts are making to the country's air defense. His great contributions through the widely used aircraft instruments which he has designed and developed during the past fifteen years are well known to every one in aviation. The opportunities for the dissemination of aeronautical knowledge which his donation to the Institute has made possible are likewise appreciated by the aeronautical profession and will be long remembered.

Donation from R. H. Fleet

Major R. H. Fleet, of San Diego, California, founder of the Consolidated Aircraft Corporation, and a Founder member of the Institute, made a gift of $10,000 to the Institute last month.

In the telegram in which Major Fleet notified the Institute of his intention to make this donation, he said: *"Being deeply interested in aeronautical science and realizing the fine work your Institute is carrying on along these lines, I desire to give $10,000 to assist it."*



R. H. Fleet

The Officers and Council at their last meeting passed a resolution expressing, on behalf of the Institute, their appreciation of Major Fleet's interest in the progress of the Institute as stated in his message and so generously demonstrated through his gift for the furtherance of its work.

Fairchild Publication Fund Established

Sherman M. Fairchild, Founder member of the Institute, Chairman of the Board of the Fairchild Aviation Corporation and the Fairchild Engine and Airplane Corporation, donated to the Institute last month a fund of $25,000 for the publication of technical papers, reports and other aeronautical material that would not otherwise be published in aviation trade magazines or scientific journals.

In its resolution accepting this gift and conveying the appreciation of the Institute to Mr. Fairchild for his generosity, the Council set it up as a fund to be known as "The Sherman Fairchild Publication Fund" and stipulated that all publications issued by it should bear this imprint. The fund will be managed by a committee consisting of Mr. Fairchild, Charles H. Colvin and Lester D. Gardner. They will be advised as to the choice of material for publication by the Editorial Board of the Journal of the Aeronautical Sciences.

It is intended that part of the fund be used in the near future to publish certain technical reports whose value, though vital to the aeronautical sciences, is not of sufficient general interest to warrant its inclusion in magazines of general circulation. It will also make possible the reproduction of engineering papers whose circulation, for reasons of military secrecy, must be restricted to the government approved list of technical people engaged in national defense work who are authorized to receive confidential material. The services of the fund have already been offered to the Army and Navy air forces for the publication and distribution of reports as they may suggest.

Major Harry G. Armstrong Named for Jeffries Award

Major Harry G. Armstrong, U.S. Army Medical Corps, in charge of research at the School of Aviation Medicine, Randolph Field, Texas, has been named to receive the John Jeffries Award given by the Institute, for outstanding contributions to the advancement of aeronautics through medical research.

Major Armstrong established the Aero Medical Research Laboratory of the Army Air Corps at Wright Field, Dayton, Ohio, in 1934 and was the director of this unit until his recent assignment to duty at Randolph Field. Born in DeSmet, South Dakota, February 17, 1899, he took pre-medical courses at the University of Minnesota, received a B.S. degree from the University of South Dakota in 1923 and his M.D. from the University of Louisville (Kentucky) in 1925. He served as an interne at St. Luke's Hospital, St. Paul, Minnesota, and is a graduate of the Army Medical School, the Army Medical Field Service School and the Army School of Aviation Medicine.

The John Jeffries Award will be conferred on Major Armstrong in recognition of his pioneering studies done on the physiological and psychological effects of flying at high altitude and in high speed maneuvers. He was one of the first to recognize and accurately describe some specific medical results of flying, such as aero-neurosis, a kind of mental and physical fatigue experienced by flyers under certain conditions; aero-otitis, an effect of high altitude and acrobatic flying on the human middle ear; and aero-embolism, a reaction similar to the "bends" suffered by deep sea divers which affects pilots who climb to high altitudes too rapidly and without proper safeguards.

Most of his researches on the physiological effects of low baro-

Sherman M. Fairchild

TENTH ANNUAL MEETING AND HONORS NIGHT DINNER

JANUARY 27–31, 1942, NEW YORK

Honors Night Dinner

HOTEL WALDORF-ASTORIA, NEW YORK

TUESDAY, JANUARY 27, 7:30 P.M.

Marking the Institute's tenth year as the only scientific aeronautical society in the United States, the Honors Night Dinner opens the sessions of the Tenth Annual Meeting in New York. Due to the large attendance at previous dinners and the even greater number expected on this anniversary occasion, more ample accommodations have been engaged for this affair than in previous years. Reservations for the dinner are $6.00 per person. The program for the Dinner follows:

Greeting to Members and Guests, by Frank W. Caldwell, President of the Institute of the Aeronautical Sciences.

Presentation of The Daniel Guggenheim Medal to Juan T. Trippe.

Presentation of The Sylvanus Albert Reed Award to Dr. Theodore von Kármán.

Presentation of The Octave Chanute Award to Melvin N. Gough.

Presentation of The Lawrence Sperry Award to Ernest G. Stout.

Presentation of The John Jeffries Award to Major Harry G. Armstrong (M.C.).

Presentation of The Robert M. Losey Award (recipient to be announced).

Presentation of Certificates to Fellows of the Institute Elected in 1941.

Presentation of Certificates of Honorary Membership in the Institute to

Hon. Robert A. Lovett, Assistant Secretary of War for Air;

Hon. Artemus L. Gates, Assistant Secretary of the Navy for Air;

Hon. Robert H. Hinckley, Assistant Secretary of Commerce for Air.

Induction of Hall L. Hibbard as President of the Institute for 1942.

Technical Sessions of Annual Meeting

PUPIN PHYSICS LABORATORIES

COLUMBIA UNIVERSITY, NEW YORK

Technical papers will be presented at morning, afternoon and evening sessions over a period of four days. Luncheon will be served each day at the Columbia University Men's Faculty Club. Since numerous changes have had to be made in the list of papers to be presented, the titles are not given here. Members of the Institute have been informed as to the papers being presented through programs sent by mail. Final program will be available at the meeting. The schedule and location of sessions is given below.

WEDNESDAY, JANUARY 28

9:15 a.m. **Structures, Part 1,** Room 401
Chairman, Eugene E. Lundquist
Power Plants, Room 428
Chairman, R. N. DuBois

2:00 p.m. **Structures, Part 2,** Room 401
Chairman, John E. Younger
Radio and Instruments, Room 428
Chairman, W. A. Reichel

5:00 p.m. **Annual Business Meeting,** Room 401

8:00 p.m. **Design,** Room 401
Chairman, Fred E. Weick

THURSDAY, JANUARY 29

9:15 a.m. **Materials, Part 1,** Room 401
Chairman, C. F. Nagel, Jr.
Rotating Wing Aircraft, Room 428
Chairman, Ralph H. McClarren

2:00 p.m. **Materials, Part 2,** Room 401
Chairman, A. V. de Forest
Physiologic Problems, Part 1, Room 428
Chairman, Arnold D. Tuttle

8:00 p.m. **Physiologic Problems, Part 2,** Room 401
Chairman, John G. Jenkins

FRIDAY, JANUARY 30

9:15 a.m. **Aerodynamics, Part 1,** Room 428
Chairman, A. M. Kuethe
Meteorology, Part 1, Room 401
Chairman, C. F. Brooks

2:00 p.m. **Aerodynamics, Part 2,** Room 428
Chairman, William Bollay
Meteorology, Part 2, Room 401
Chairman, F. W. Reichelderfer

8:00 p.m. **Air Transport,** Room 401
Chairman, Charles Froesch

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metric pressure, low temperature and lack of oxygen were carried out in the high altitude test chamber of the Aero Medical Research Unit at Wright Field. Major Armstrong customarily made himself the first human subject for a new test, often under experimental conditions which were dangerous to his own life. His work has done much to stimulate further research in the medical aspects of aviation and has led to the development of oxygen supply and pressurizing apparatus and other precautionary devices and procedures for safeguarding pilots against the physical effects of modern military flying. Numerous reports on his work have been published in medical and aeronautical journals. His book on "Principles and Practice of Aviation Medicine" is the most complete and authoritative text yet published on the subject.

Major Armstrong is a member of the Aero Medical Association, the Association of Military Surgeons and the American Medical Society. He is an Associate Fellow of the American College of Physicians and of the Institute of the Aeronautical Sciences. He was chosen to receive the John Jeffries Award for 1941 by a committee consisting of Dr. G. W. Lewis, Director of Research of the National Advisory Committee for Aeronautics; Gill Robb Wilson, President of the National Aeronautical Association; Edgar S. Gorrell, President of the Air Transport Association of America; Capt. J. R. Poppen, U.S.N., President of the Aero Medical Association; Frank W. Caldwell, President of the Institute of the Aeronautical Sciences; and Lester D. Gardner, President of the Aeronautical Archives of the Institute.

The award is named after Dr. John Jeffries, a Boston physician, who was the first American to make scientific observations from the air. He made a flight from London with Blanchard, the French balloonist, in 1784 and took recordings of the temperature, humidity and density of the air. In January, 1785, Dr. Jeffries and Blanchard made the first crossing of the English Channel by air.

Lawrence Sperry Award for 1941 to Ernest G. Stout

The Lawrence Sperry Award for a notable contribution made by a young man to the aeronautical sciences will be presented to Ernest G. Stout at the Institute's Honors Night Dinner in New York on January 27. The award consists of an honorarium of $250 and a certificate which will bear the citation "For his contributions to the experimental determination of the hydrodynamic stability of model flying boats and seaplanes."

Mr. Stout, a MEMBER of the Institute, is Engineer in charge of Aerodynamics and Flight Test for Consolidated Aircraft Corporation. He was an Aerodynamic Engineer for this company from 1936 until 1938, then Engineer in charge of Hydrodynamics until he recently assumed his present position.

Born in Moran, Kansas, April 25, 1913, he graduated from Western High School, Washington, D. C., in 1931, and received The Perry Collins McDonough Scholarship for four years at the Daniel Guggenheim School of Aeronautics, New York University. He received the degree of B.S. in Mechanical Engineering from New York University in 1935 and served as a Research Assistant there for three months in the summer of that year. In 1939 Mr. Stout received a non-resident degree as Aeronautical Engineer for which he submitted a thesis entitled "Charts for the Determination of Seaplane Performance."

He was an Aerodynamic Engineer with The Glenn L. Martin Company from July, 1935, until November, 1936, when he joined the engineering staff of Consolidated.

Mr. Stout's work on the hydrodynamic stability of flying boats and seaplanes, for which he receives the award, was begun in 1938. He read a general paper on "Notes on the Design and Stability of Flying Boats" before the Los Angeles Section of the Institute in September, 1939, and his report on "Experimental Determination of Hydrodynamic Stability" was published in the Journal of December, 1940. These described his design of seaplane and flying boat models with controllable elevator angle and certain modifications on wing and tail surfaces to counteract scale effect. The methods of mounting and testing these models as supported from a boat in open water or from the carriage in a towing basin, have permitted the determination of the effect on stability of the center of gravity location and hull bottom modifications. Thus the changes in hull design and construction necessary to prevent dangerous porpoising can be made and tested in small models before being incorporated in a full size flying boat. Mr. Stout's methods have been used by three government departments and several manufacturers and have been closely checked by full scale tests.

The Award, honoring the memory of Lawrence Sperry, pioneer aviator and inventor who died in 1923 at the age of 31, is endowed with a fund of $10,000 by his brothers and sister.

The recipient of The Lawrence Sperry Award is chosen by a Board of Award whose members in 1941 were Lt. Col. James H. Doolittle, Lester D. Gardner, Charles L. Lawrance, Grover Loening, Glenn L. Martin and Elmer A. Sperry, Jr.

News of Institute Members

Eli Amaneul, Technical Member, has joined the staff of the Munk Aeronautical Laboratory as a member of the firm.

Albert A. Andrake, A.F.I.Ae.S., Project Tool Engineer, The Glenn L. Martin Co., for the past two years, has been transferred to The Glenn L. Martin-Nebraska Co., Omaha, Nebraska, as Chief Production Engineer.

W. John Beil, Technical Member, is now in the Aerodynamics Section, Airplane Division, Curtiss-Wright Corp., Buffalo, N. Y.

Raymond L. Bisplinghoff, Technical Member, formerly Research Assistant, has become Instructor in Aeronautical Engineering at the University of Cincinnati.

Allan F. Bonnalie, A.F.I.Ae.S., Assistant to United Air Lines' Executive Vice-President in charge of operations, has taken a one year leave of absence effective since Dec. 1, to join the Bureau of Aeronautics, Navy Dept. He is a Lt. Comdr. in the U.S. Naval Reserve.

Douglas Campbell, M.I.Ae.S., has been named Vice-President of Pan American-Grace Airways. He has been active in Panagra since May, 1935.

Robert Alden Cornog, Technical Member, who has been a Research Fellow at the Univ. of California, is now a Physicist, Bureau of Ordnance, Navy Dept.

Peter de Florez, Technical Member, has left the Ranger Aircraft Engines Div. of Fairchild Engines & Airplane Corp. to join the Link Aviation Devices at Binghamton, N. Y.

Albert B. Deyarmond, M.I.Ae.S., is now a Captain in the Army, Specialist Reserve U.S. Army Air Corps, at Wright Field. He was formerly Chief of the Stress Dept., Vultee Aircraft, Inc.

Lt. Col. James H. Doolittle, F.I.Ae.S. and Past President of the Institute, has been promoted to this rank recently, it was announced by the War Department. Formerly Director of Aviation for Shell Oil Company, he has been on active duty since July, 1940, as a Major in the Procurement Division of the Army Air Corps. From November of that year he was stationed at Detroit and is now engaged on research and engineering problems pertaining to aircraft production, with headquarters at Baltimore.

Ralph H. Draut, M.I.Ae.S., has been named Chief of Structures, The Glenn L. Martin-Nebraska Co. Formerly a Project Stress Engineer for The Glenn L. Martin Co., he has been associated with such firms as the Curtiss Aeroplane & Motor Co., Stearman Aircraft Co., and Waco Aircraft Co.

John C. Dykes, Technical Member, formerly the Assistant to the Research Engineer of British Overseas Airways Corp. at Bristol, England, has been transferred to Asmara, Italian East Africa (British Occupied). He is a Technical Officer at the company's base there.

Reuben H. Fleet, M.I.Ae.S., having sold his interest in the Consolidated Aircraft Corp., which he founded, to Vultee Aircraft, Inc., has resigned as President and Director of the company. He will remain with Consolidated in an advisory capacity.

A. Milton Drummer, Technical Member, is now a Stress Analyst with the Beech Aircraft Corp.

George L. Fruauff, Jr., Technical Member, recently with the Brewster Aeronautical Corp., has become associated with the Western Electric Co. at Kearney, N. J.

R. E. Gillmor, A.F.I.Ae.S., President of the Sperry Gyroscope Co., has become a member of the Board of Directors of the Fairchild Engine and Airplane Corp.

George M. Goldman, Technical Member, has joined the Materiel Division, U.S. Army Air Corps, Wright Field.

Nicholas J. Hoff, A.F.I.Ae.S., Assoc. Professor of Aeronautical Engineering at the Polytechnic Institute of Brooklyn, has been elected an Associate Fellow of the Royal Aeronautical Society.

Roger Wolfe Kahn, M.I.Ae.S. and member of the Council, has been appointed a member of the Aviation Planning Staff, a research and advisory body attached to the Civil Air Patrol.

Major Reed G. Landis, M.I.Ae.S., of the Office of Civilian Defense, which organized the Civil Air Patrol, is head of the Planning Staff.

Herbert F. Kueck, Technical Member, formerly with the Beech Aircraft Corp., is now an Instructor at the Guggenheim School of Aeronautics, N. Y. U.

Harvey A. Leich, Technical Member, recently with the Grumman Aircraft Engineering Corp., is now a 2nd Lt. and Engineering Officer with the First Marine Aircraft Wing, U.S.M.C. Reserve.

Richard C. du Pont, M.I.Ae.S., President of All American Aviation, Inc., of Wilmington, Del., has announced that **Grover Loening,** F.I.Ae.S., aircraft manufacturer and engineer, has become associated with the company as Engineering Advisor, and also has been elected a member of the Board of Directors.

Cameron N. Lusty, Technical Member, has been named Chief Engineer of the Mercury Aircraft Co. He was formerly a Stress Analyst at the Taylorcraft Aviation Corp.

Victor J. Martin, M.I.Ae.S., has joined Vultee Aircraft, Inc., as an Aerodynamics Research Engineer. For the past four years he has been teaching aeronautical engineering at the University of Washington and at the time of his resignation was an Asst. Professor.

Robert A. McConnell, Technical Member, is now a Research Associate, Radiation Laboratory, Mass. Inst. of Technology.

Harold P. Moon, M.I.Ae.S., formerly General Manager, Summit Aeronautical Corp., is now a test pilot of military aircraft, Airplane Div., Curtiss-Wright Corp., Buffalo, N. Y.

Robert R. Osborn, F.I.Ae.S., is now Asst. Engineering Advisor of the Bell Aircraft Corp. A graduate of the Univ. of Pennsylvania, Mr. Osborn has been with the Curtiss Aeroplane & Motor Co., and more recently was Asst. Director of Engineering, Vultee Aircraft, Inc.

Kingsbury E. Parker, Jr., Technical Member, has been made a Captain in the Royal Air Force Ferry Command, Montreal.

C. Kendall Perkins, A.F.I.Ae.S., has left American Airlines and is now a Project Engineer with the McDonnell Aircraft Corp.

Capt. John R. Poppen, U.S.N., M.I.Ae.S., has been transferred from the Bureau of Aeronautics, to the Bureau of Medicine and Surgery, Navy Dept.

Allen E. Puckett, Technical Member, is now a Research Assistant, Calif. Inst. of Technology.

E. Robert Reff, M.I.Ae.S., formerly Structural Engineer at the Boeing Aircraft Co., is now Structural Engineer with the Aircraft Div. of Fairchild Aircraft Corp. at Hagerstown, Md.

Peter F. Rossmann, M.I.Ae.S., formerly Development and Research Engineer, has been named Assistant to the Director of Military Engineering, Airplane Div., Curtiss-Wright Corp., Buffalo, N. Y.

Alexander Satin, M.I.Ae.S., has joined the Consolidated Aircraft Corp. as Design Engineer. He was formerly Chief Engineering Instructor, School of Aeronautics, California Flyers, Inc.

Richard A. Schram, Technical Member, has been named Senior Ground School Supervisor, CAA, in the Chicago area.

Comdr. C. H. Schildhauer, U.S.N., A.F.I.Ae.S., recently Marine Equipment Sales Manager for The Glenn L. Martin Co., reported for active duty with the Navy on December 1.

N. F. Silsbee, M.I.Ae.S., aviation writer and lecturer, and formerly Comptroller of Beaver College, Jenkintown, Pa., has been on active duty since October as a Captain in the Air Corps. He is Chief of the Research Section, Public Relations Branch, Hq. Army Air Forces.

Lt. Emil L. Sorenson, Technical Member, is now a Test Pilot, Materiel Div., U.S. Army Air Corps, Wright Field.

Morril B. Spaulding, Technical Member, has joined The Glenn L. Martin Co. as a Project Liaison Engineer. He was formerly with the Pan American Airways System as a Jr. Engineer.

John F. Vogel, Technical Member, has been called to active duty as a 1st Lieut., Signal Corps, MacDill Field, Florida. He was formerly with the Stinson Aircraft Div. of Vultee Aircraft, Inc.

Capt. W. W. Webster, U.S.N., F.I.Ae.S., has recently been transferred from the office of the Material Branch of the Navy Bureau of Aeronautics in Washington to become Manager of the Naval Aircraft Factory in Philadelphia.

Robert B. White, M.I.Ae.S., formerly Asst. to Vice-Pres., TWA, Inc., is now Factory Manager, North American Aviation, Inc., of Kansas.

Robert S. Wolfsohn, Technical Member, now assigned to a project at the National Bureau of Standards in Washington, is an Assistant Mechanical Engineer on the staff of the N.A.C.A.

Bradley H. Young, Technical Member, who has been connected with the Maintenance staff of Pan American-Grace Airways since 1937, has been appointed Maintenance Engineer at Lima, Peru.

Shao W. Yuan, Technical Member, has joined the Vidal Research Corp. as an Aeronautical Engineer.

Harry Zuckerberg, M.I.Ae.S., is now Chief of Research and Development, Platt-LePage Aircraft Co. He was formerly Chief Structures Engineer.

Personnel Opportunities

The Personnel Bureau serves individual members, as well as organizations seeking to employ aeronautical specialists. Any member or organization may have requirements listed without charge.

Available

Aeronautical and Mining Engineer, familiar with Air Corps and Bureau of Aeronautics contract procedure desires an opportunity to make new connection. Continuously employed for approximately 15 years on production engineering work involving production schedules, materials and other aspects of procurement planning incident to aircraft production. Address reply to Box 146, Institute of the Aeronautical Sciences.

Student Branches

Aeronautical University. A paper on "The Workings and Uses of the Terrain Clearance Indicator and the Flightray" was presented by Edward Bedrosion at a meeting held on December 2.

Indiana Technical College. Two films, borrowed from the Institute, were shown at a meeting held on December 2. They were "Conquest of the Air" and "Goodrich De-Icer Development."

Massachusetts Institute of Technology. W. A. Petrasek, of American Airlines, gave a talk on the Sperry "Flightray" at a meeting held on December 9. At a previous meeting, Thomas D. Perry, of The Resinous Products & Chemical Company, gave a talk and demonstration of aircraft plywoods.

University of Pittsburgh. At the first meeting of this new Student Branch, officers were elected as follows: Marshal Nesbitt, Chairman; Joseph Siro, Vice-Chairman; Raymond Zavasky, Treasurer; Theodore Just, Secretary. J. Elwood Arnold is Faculty Advisor.

Rensselaer Polytechnic Institute. At a meeting held on December 11, two U.S. Army Air Corps films were shown: "Training of Air Corps Cadets" and "Wings of Steel." Later in the evening, the present officers of the Branch were re-elected. They are: George McTigue, Chairman; Warren Tucker, Treasurer; and Richard P. Graham, Secretary.

Changes in Membership

The following changes in the membership of the Institute have occurred since the publication of the previous issue of the Journal.

Transferred to Associate Fellow Grade

Blount, Earl Ellsworth, Chief of Structures, Airplane Div., (St. Louis Plant), Curtiss-Wright Corp.

Doman, Carl Templar, B.S. in E.E.; Vice-Pres. in charge of Engineering, Aircooled Motors Corp.

Hoff, Nicholas John, Dipl. Ing.; Assistant Professor of Aero. Eng., Polytechnic Institute of Brooklyn.

Elected to Member Grade

Barton, Millard Vernon, Ph.D.; Asst. Prof. of Mech. Engineering, University of Maryland.

Goodwin, Ralph Talbot, Ph.D.; Manager, Aviation Dept., Shell Oil Co., Inc.

Greer, Edward Maurice, B.S. in Ae.E.; Chief Design Engineer, Simmonds Aerocessories, Inc.

Horrisberger, John Frederick, B.S. in M.E.; Project Engineer (Airplanes), U.S. Naval Aircraft Factory, Philadelphia.

Howard, Henry Bernon, B.A. in M.E.; B.Sc.; Assistant Director, British Air Commission.

Kelly, Everett Lowell, Ph.D.; Associate Professor of Psychology, Purdue University (See A. M. of S.).

Krzywoblocki, Maria Zbigniew, Dipl. Ing.; M.S.; A.F.R.Ae.S., Asst. to Supervisor of Tool Control & Works Method Dept., Aircraft Div., Canadian Car and Foundry Co.

Shaw, William Ulric, B.A.Sc.; Chief Engineer, Aircraft Div., National Steel Car Corp., Ltd., Canada.

Watson, Donald Reed, Chief Weight Engineer, Fleetwings, Inc.

Wiggins, Garwood Owen, Executive Engineer, Ranger Aircraft Engines Div., Fairchild Engine & Airplane Corp.

Transferred to MEMBER Grade

Allen, Fred Charles, M.E. in Ae.; Ass't. Chief of Stress, El Segundo Div., Douglas Aircraft Co.

Walker, Ernest E., A.B.; M.L.A.; Aeronautic Constructor for Airports and Airways.

Elected to Industrial Grade

Zweng, Charles Alfonso, LL.B.; Owner and Manager, Pan American Navigation Service.

Elected to Technical Member Grade

Anderson, Benjamin Monroe, B.A.; Detail Engineer, Boeing Aircraft Co.

Bates, William Elmer, Jr., B.S. in M.E.; Technical Engineering Assistant, Blade Design, Curtiss Propeller Div., Curtiss-Wright Corp.

Ebeling, Wilbur Ludlow, B.S. in Ae.E.; Instructor, Wayne University.

Greenewald, Raymond Eugene, B.S. in Ae.E.; Instructor Aero. Dept., Tri-State College.

Hermes, Raymond Francis, B.Ae.E.; Stress Analyst, Aeronca Aircraft Corp.

Ill, Edward Joseph, II, Weight and Balance Engineer, AGA Aviation Corp.

Lentz, George Lawrence, B.S. in E.E.; Engineer, General Electric Co.

Meyer, Carl George, Engineer, Republic Aircraft Corp.

Neeley, Glen E., Assistant Instructor, Post Schools Dept., U.S. Army Air Corps, McClellan Field, Calif.

Pfeiffer, Shelly Bernhardt, B.S. in M.E.; Stress Group, El Segundo Div., Douglas Aircraft Co.

Schnaubelt, Franz Joseph, Planning Engineer, Group Leader, S. I. Group, Production Engineering Dept., Consolidated Aircraft Corp.

Schell, Charles Joseph, Jr., Engineering Instructor, Casey Jones School of Aeronautics.

Westergaard, John William, Engineer, Boeing Aircraft Co.

Transferred from Student Member to Technical Member Grade

Alexander, Francis Brown, B.S. in M.E.; 2nd Lt. A.C., Air Corps Tactical School.

Belzer, Maurice, B.S. in Ae.E.; Engineer, Vought-Sikorsky Aircraft Div., United Aircraft Corp.

Bemer, Robert W., A.B.; Engineering Calculator, Douglas Aircraft Co.

Beven, Kenneth A., Engineering Draftsman, El Segundo Div., Douglas Aircraft Co.

Bielat, Ralph Philip, B.S. in Ae.E.; Jr. Aero. Engineer, Langley Memorial Lab., N.A.C.A.

Blake, Robert W., S.B. in Ae.E.; Member Engineering Dept., Pan American Airways, Inc.

Bogert, Robert C., B.S. in Ae.E.; Aviation Cadet, U.S. Army Air Corps.

Bonney, Robert Wm., Inspector, Consolidated Aircraft Corp.

Brogger, Charles, C.B.S. in Ae.E.; Draftsman, Designer, Experimenter, Camfield Mfg. Co.

Buie, John C., Instrument Technician, Mackenzie Air Service Ltd.

Burgeson, John Edward, Draftsman, Ryan Aeronautical Co.

Castiglione, Frank P., B.S. in Ae.E.; Engineering Draftsman, Dept. of Commerce.

Chin, Charles, B.S. in Ae.E.; Loftsman, Airplane Div. (Columbus Plant), Curtiss-Wright Corp.

Christie, Sidney A., Draftsman, Ryan Aeronautical Co.

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Dilworth, John A., B.S. in Ae.E.; Lt. U.S. Army Air Corps.

Dugan, James C., A.E.; Jr. Aero. Engineer, U.S. Army Air Corps.

Frei, Ralph E., Aero. Engineer, The Glenn L. Martin Co.

Glossa, Milton, B.S. in Ae.E.; Detail Draftsman, Boeing Aircraft Co.

Harvey, Jacque W., B.S. in Ae.E.; Stress Analyst, Bell Aircraft Corp.

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Hughes, George W., Jr., M.E.; Liaison Engineer, Consolidated Aircraft Corp.

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Kantrow, Harold, A.B.; Scientific Aide, Langley Memorial Aero. Lab., N.A.C.A.

Lamar, William Edgar, B.S. in Ae.E.; Lt. U.S. Army Air Corps.

Lewis, Richard B., B.S. in Ae.E.; Flight and Apprentice Engineer, Pan American-Grace Airways, Inc.

Lorenz, Sigmund J., Drafting, The Glenn L. Martin Co.

Mayer, Norman J., Aero. Engineer, Goodyear Aircraft Corp.

McMillan, William P., B.S. in Ae.E.; Student Instructor, U.S. Army Air Forces.

Moore, Wesley Dean, Layout Draftsman, El Segundo Div., Douglas Aircraft Co.

Mortlock, Walter E., B.S. in Ae.E.; Power Plant Analysis Engineer, Boeing Aircraft Co.

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Speer, Edmund Magnus, Jr., B.S. in Ae.E.; Test Inspector, Pratt & Whitney Aircraft Div., United Aircraft Corp.

Strickland, John T., M.S. in Ae.E.; Asst. Aero. Engineer, U.S. Army Air Corps.

Swanson, Douglas.

Tenedini, A. S., Student, Casey Jones School of Aeronautics.

Vondenberg, Joseph L., B.S. in Ae.E.; Private, U.S. Army.

Wall, Thomas F., Draftsman, Douglas Aircraft Co.

Warshaw, Seymour, B.C.E.; Jr. Naval Architect, U.S. Navy Yard, Philadelphia.

Winkler, Karl J., B.S. in Ae.E.; Jr. Engineer, Curtiss Propeller Div., Curtiss-Wright Corp.

Wisely, Joseph A., B.S. in Ae.E.; Jr. Engineer, Curtiss Propeller Div., Curtiss-Wright Corp.

Necrology

Hart O. Berg

Hart O. Berg, Benefactor Member of the Institute, died in New York on December 9, 1941, after an illness of several weeks.

Due to his early connection with the work of the Wright brothers and his continued interest in aviation, Mr. Berg had kept a large collection of documents on their work. In 1939 he presented this collection to the Aeronautical Archives of the Institute. They contain the most complete first hand record of the early flights of the Wright Brothers in Europe.

Hart O. Berg was born in Philadelphia, on March 23, 1865. He received an engineering education in Liege, Belgium, and worked for several years in a Belgian gun plant. He accepted, in the early nineties, the position of Assistant Manager of the Colt Patent Fire Arms Company's factory at Hartford, Conn. During his two years at the Colt plant the manufacture of the old Gatling gun was discontinued and the new Browning machine gun was developed under his direction.

He then went back to Belgium where for six years he was successively Assistant Manager and General Manager of the Fabrique Nationale d'Armes de Guerre at Herstal, Liege (Belgian National Gun Works), building Mauser rifles. He brought John Browning to Liege and manufactured the first Browning pistols at the Belgian plant before they were produced commercially in the United States. He also traveled extensively and obtained important gun and munition contracts in almost every country of the world for the Fabrique Nationale.

In 1899 he became Chief Engineer for Clement-Bayard in Paris, and erected there the world's first large automobile plant. He represented the United States as American Member of the Machinery Jury at the Paris Exposition in 1900.

In 1902 he organized the Berg Automobile Company and built the "Berg," one of the first large four cylinder cars constructed in America.

Shortly afterward he became associated with Captain Simon Lake, and obtained, early in 1904, an order from the Russian Government for six submarine boats. He spent three years in Russia building ten submarines for the Russian Navy, the last four being of a larger type than any previously built, each mounting for the first time a large calibre gun. He also built, with the late Lewis Nixon, ten torpedo Boats at Sebastopol. Afterward he built, with Simon Lake, two submarine boats in Pola for the Austrian Government.

In 1908 Mr. Berg brought Wilbur Wright to Europe and hired a race course near Le Mans, France, where in September of that year Wright made his first public airplane flights and so started practical aviation. He then organized in France the world's first aviation company. Shortly afterward Wilbur Wright won the Michelin Prize of 20,000 francs (then $4000) for directed flight over a line of toy balloons 100 meters high. He organized the first flying field at Pau and later a flying field outside of Rome, and the following year took Orville Wright to Germany where the first airplane flights in Germany took place at the Tempelhof, Berlin. During this period he also brought to France Lieutenant Reilly-Scott, a West Point graduate, who won the first Michelin Prize for dropping bombs at a target from an airplane.

In 1910 he became interested in a new method of obtaining sulphur from waste gases in smelter operations, and shortly afterward was instrumental in erecting the first oil cracking plant in Europe at the mouth of the River Thames to increase production of gasoline.

Later during World War I he designed, equipped, and installed ten ice plants for the making and distributing of ice to Field Hospitals on the Eastern Front.

Since the War he was employed in metallurgical operations principally concerning the production of metallic magnesium and its commercial introduction.

He had made Paris his home for many years up to 1939 and had his main offices and staff for general engineering and consulting work there as well as branch offices in London and New York. He made more than 146 trips across the Atlantic, besides encircling the world several times.

Mr. Berg was a member of the American Society of Mechanical Engineers, of which he was Honorary Vice-President and their representative at the Memorial Ceremonies in honor of the French scientist Le Chatelier, member of the Engineers Club, New York, one of the Founders in 1898 of the Aero Club de France, also of the Automobile Club de France, member of the American Club of Paris, one of the Founders of the St. Cloud Country Club, member of the Royal Automobile Club, London, and one of the first members of the American Chamber of Commerce in Paris.

He was decorated twice by the French Government; in 1901 as Chevalier, and in 1909 as an Officer of the Legion of Honor.

Tullio Levi-Civita

Dr. Tullio Levi-Civita, foreign Associate Fellow of the Institute, died in Rome, Italy, on January 1, 1942, at the age of 69.

He was one of the small group of distinguished foreign scientists invited to be among the Founder Members of the Institute. Dr. Levi-Civita was regarded as Italy's greatest mathematician and had done important work in the fields of hydrodynamics, theoretical dynamics and pure geometry. His contribution to the theory of absolute differential calculus was said to have helped make possible Einstein's general theory of gravitational relativity. His concept of parallelism in higher spaces was regarded as showing the way to the development of the new differential geometry.

Dr. Levi-Civita was educated at the University of Padua, from which he received a Doctorate in Mathematics in 1894. He had served as Professor of Rational Mechanics at the University of Rome since 1919. He was a member of the Pontifical Academy of Sciences, the Associazione Aerotecnica Italiana and a foreign member of the British Royal Society.